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RESULTS OF RELATIVITY WITHOUT THE THEORY OF TENSORS.*

BY R. A. M. KEARNEY.

THE Special Theory of Relativity leads to the following conclusions :

(i) The three dimensions of space and one of time constitute an isotropic fourfold, in which there is no unique time-direction, just as there is no unique space direction.

(ii) Specifying the velocity of a body is equivalent to specifying its "time-direction", that is, the direction of its path through space-time (its "world-line").

(iii) If the units of distance and time are taken to correspond with each other, so that the velocity of light is unity, then the lines in our diagram ($L'OL$, $M'OM$, and lines parallel to these) which represent the paths of light pulses (Robb's "optical lines") are at right angles for light going in opposite directions, and in this case the time direction ($T'OT$) of a particle and its corresponding space-direction ($X'OX$) are equally inclined to the optical line ($L'OL$) between them, so that setting a particle in motion involves "rotating" its time-direction and its space-direction through equal angles (T_1OT_2, X_1OX_2) towards the optical line which goes in the direction of motion of the particle.

(iv) The curve of constant radius and uniform curvature in a time-distance plane (acceleration plane) instead of being a circle, as in an ordinary space-plane, is a rectangular hyperbola having a pair of optical lines as asymptotes. (Thus, in the figure, $OT_1 = OT_2$, and $OX_1 = OX_2$). Unlike the circle, this curve has a circumference of infinite length.

* In the original form of the paper, as prepared for the Annual Meeting of the Mathematical Association on 4th January, 1934, the argument was based on the consideration of matter as a process, without assuming the Principle of Relativity or Laplace's Equation. This being too lengthy for inclusion in the *Gazette*, the more fundamental parts of the paper have been omitted, and the argument modified so as to make use of the results of the Special Theory of Relativity and to obtain the new law of gravitation as a natural extension of Laplace's Equation.

(v) Relative velocity is not additive, but if, instead of relative velocity, we take the "hyperbolic angle" between the world-lines of particles, then this quantity (called by Robb the "rapidity") is additive for particles going in the same direction.

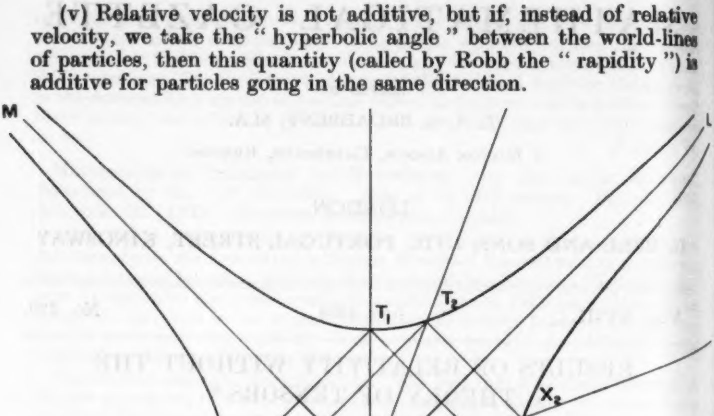


FIG. 1.

(vi) The invariant quantity connecting two points (or "moments") in the space-time fourfold, corresponding to the square of the distance between two points in space, is

$$S^2 = T^2 - X^2 - Y^2 - Z^2. \quad \text{.......................(1)}$$

It requires "rotation" through an infinite hyperbolic angle to turn any space- or time-direction into the direction of an optical line; in other words the "rapidity" of light is infinite.

In interpreting the space-plane diagram, which represents on paper the time-distance plane, we have to bear in mind that the true interval represented by a given length in our diagram depends on its slope, being less (in proportion to its apparent length) the nearer

its direction is to that of an optical line: also that hyperbolic rotation of a figure (so as not to change its *true* form) involves rotating its space-directions and their corresponding time-directions by equal amounts in *opposite* senses. (In our diagram the rectangles $T_1X_1T_1'X_1'$ and $T_2X_2T_2'X_2'$ have the same true form, as have also the triangles OX_1T_1 and OX_2T_2).

Since the velocity of a particle merely gives the direction of its path in space-time, acceleration corresponds to the curvature of its path, and is to be measured by the hyperbolic angle through which the path turns per unit of its length, that is, per unit of time *as measured by the particle*. If we use the time as measured by a body which does not move with the particle, we arrive at a wrong measure of the curvature, that is, of the acceleration, and hence our estimate of the apparent mass (that is, inertia, or resistance to acceleration) differs from the true mass. This, of course, is the explanation of the apparent increase in the mass of a particle moving with very high velocity.

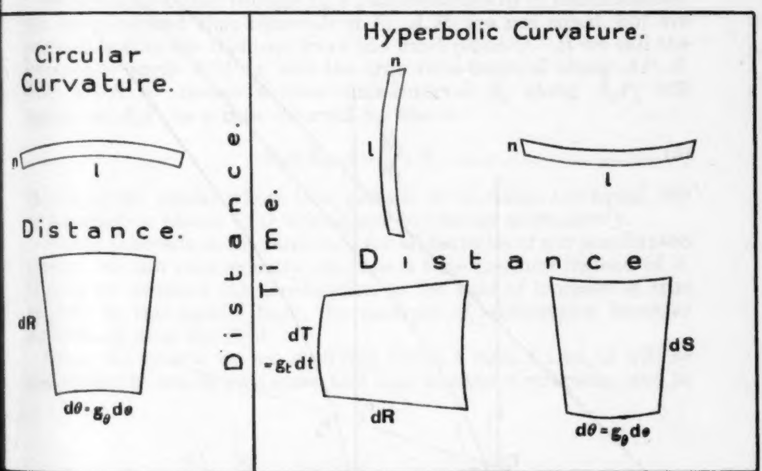


FIG. 2.

If in travelling along a curve we carry with us a normal to the curve, of constant length, then the rotation of the direction of the path is equal to the rotation of the normal. Hence, while the foot of the normal travels a distance l_0, the summit of the normal travels a distance l, where

$$l = l_0 - n\theta \text{ for circular rotation, } \dots\dots\dots\dots\dots\dots (2a)$$

$$l = l_0 + n\psi \text{ for hyperbolic rotation. } \dots\dots\dots\dots\dots (2b)$$

Hence, if l be the length of any one of these parallel curves at a

distance n from the original path, then

$$\theta = -\frac{\partial l}{\partial n}, \text{ ....(2c), and } \psi = +\frac{\partial l}{\partial n}. \text{ ....(2d)}$$

Hence the measure of

$$\text{(1) Circular curvature is } C_c = -\frac{1}{l}\frac{\partial l}{\partial n}; \text{ ....(2e)}$$

$$\text{(2) Hyperbolic curvature is } C_h = +\frac{1}{l}\frac{\partial l}{\partial n}. \text{ ....(2f)}$$

If C_h is constant, then $\frac{1}{l}\frac{\partial l}{\partial n}$ is independent of l, whence

$$\frac{\partial l}{\partial n} = l \, . f(n). \text{ ....(2g)}$$

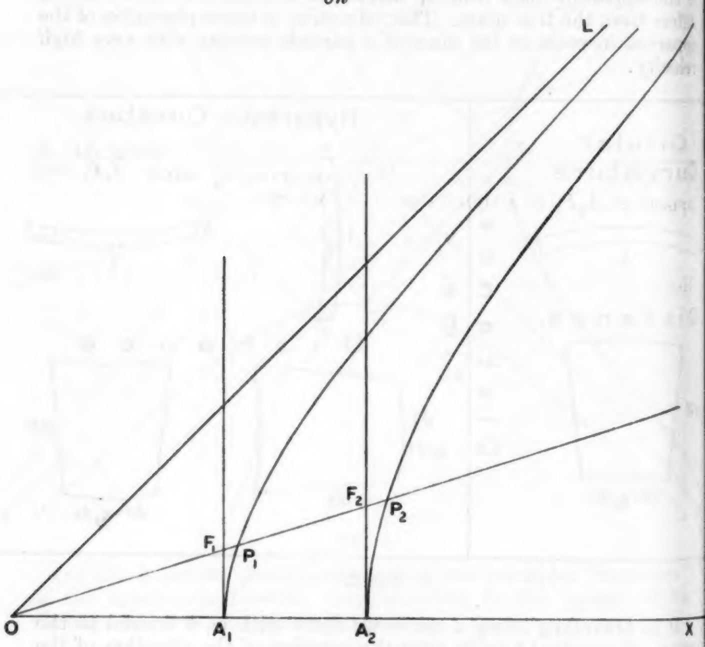


FIG. 3.

UNIFORM ACCELERATION.

Suppose a particle to start from a moment OX (Fig. 3) at a point-instant A to move with uniform acceleration in the space-direction X. Its path in space-time will be a line of uniform curvature, that is, a hyperbola having a pair of optical lines (one of which, OL, we have shown in the diagram) as its asymptotes. Referred to its

centre O, the equation of the hyperbola is $X^2 - T^2 = R^2$, where R is the constant radius of the hyperbola. It is evident that the limiting speed of the particle will be the speed of light, which corresponds to infinite rapidity.

Different values of the constant R will give different members of a system of concentric hyperbolas. If P be any point on one member of the system, OP is constant, and hence, on a radius OP_1P_2, P_1P_2 is constant. Now the radius, being normal to the curve, represents the space of the particle. Hence all the particles have a common space, and remain at constant distance apart in this space. Hence the system of hyperbolas represents the paths of particles of a rigid body moving with uniform acceleration.

We notice that the curvature of any member of the system equals $1/R$, which is different for different members ; hence the acceleration of different particles of the body is different, and varies inversely as the distance from a fixed plane (represented in our two-dimensional diagram by the point O).

We also notice that though P_1, P_2 are reckoned to be simultaneous, yet the measured time-intervals A_1P_1, A_2P_2 are not equal, but are proportional to the distance from the fixed plane O. If we call the hyperbolic angle AOP, s, and the true time-interval along AP, S, then $S = R \, . \, s$. Hence a true time-interval S_1 along A_1P_1 will appear at A_2P_2 as a time-interval S_2, where

$$S_2 = R_2 \, . \, s = \frac{R_2}{R_1} \, . \, S_1. \quad \text{......................}(3)$$

Hence of two atoms whose true periods of vibration are equal, the atom which is nearer to O will *appear* to vibrate more slowly.

Since s proceeds at the same rate for all particles of our accelerated system, we can conveniently use it as a time-measure instead of S. If then we measure the acceleration as the rate of increase of true velocity in this special time, the measure of acceleration becomes uniform all over the field.

Since the system we are studying forms a rigid frame, it will be convenient to use its own space and time as axes of reference, and to

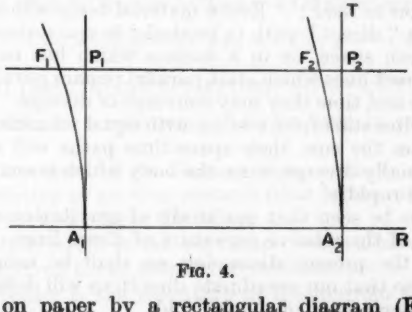


FIG. 4.

represent it on paper by a rectangular diagram (Fig. 4). The quantity s will become our time coordinate, but we shall now call it t.

Since the time and space we are using belong to the same rigid body, they are appropriate to each other, and hence all the coordinate space-directions will be normal to the coordinate time-direction. Our rectangular diagram conveniently reminds us of this.

In interpreting this diagram we have to bear in mind:

(1) that lines which appear straight may really be curved, and *vice versa*;

(2) that lines (including parallel lines) which appear equal may really be unequal, and *vice versa*;

(3) that optical lines will no longer appear to cross at right angles, or at any constant angle; the "coordinate velocity" of light will vary from point to point of the field.

Consider now the path of a free (that is, unaccelerated) particle starting from rest in our accelerated system at A. In Figure 3 it will be a straight line AF tangent to AP. In Figure 4 it will be a curve bending back in the direction $-R$. All free particles will have this negative acceleration relatively to our system. Hence we have simulation of a gravitational field.

In a true gravitational field, just as in the artificial field here considered, the natural acceleration of a free body is quite independent of its mass, or of the material of which it is composed, so that the question arises whether the natural gravitational field may be considered as due to a constraint on the motion of our frame of reference, so that gravitational force becomes identified with inertia. Since the only obvious constraint is that due to the impenetrability of matter, which must give rise to a repulsive force on immediate contact or very close approach, the inertial force brought into play must appear as an attraction; which of course agrees with experience. Since, however, this "attraction" acts radially towards the attracting body, instead of parallel towards a fixed plane, it is clear that we cannot "simulate" it with a constrained acceleration applied to a rigid body occupying *Euclidean* space. In order to identify gravitation with inertia, we shall have to assume that space-time is naturally non-euclidean, and modify Newton's First Law of Motion to read: "Every material body which is free to do so traverses a "direct" path (a geodesic) in space-time".

In Euclidean space, or in a surface which has no "inherent" curvature, direct lines which start parallel remain parallel, but in the natural space and time they may converge or diverge. For instance, if two meteorites start from rest (or with equal velocities) at different distances from the sun, their space-time paths will start parallel, but will gradually diverge, since the body which is nearer to the sun will fall more rapidly.

It will thus be seen that our study of gravitation resolves itself into a study of the relative curvature of direct lines in space-time. Throughout the present discussion we shall be using orthogonal coordinates, so that our coordinate directions will define an "orientation" at every point of the fourfold.

The curvature of a coordinate line will then be the rate of rotation

of this orientation, in a plane containing the line, as we proceed along the line. We can measure this curvature by formulae (2), and hence, by integration round a contour, we can find how much an orientation appears to rotate on travelling round such a contour. This net rotation for a contour may be called the "angle discrepancy" of the area enclosed by the contour. For a Euclidean plane the angle discrepancy will be zero. The angle discrepancies of adjacent areas will clearly be additive; hence we can find the average angle discrepancy per unit area. For a vanishing area including a particular point this will be called the "inherent curvature" of the surface at the point considered.

Referring to Figure 3, we see that, for the simple acceleration field so far considered, the velocities of two free particles which start from rest at the same *coordinate* moment OAX will remain equal at the end of equal *coordinate* intervals of time. That is to say, the hyperbolic angles at F_1 (interior) and F_2 (exterior) of the quadrilateral $A_1A_2F_2F_1$ are equal. Since also the sides of the quadrilateral are all straight, its angle discrepancy is zero. This must hold for all areas in the plane XOL. Since this property does not hold in a natural field, a natural "acceleration plane" in a gravitation field will have inherent curvature.

In what follows C^Y_{XZ} is the rate of rotation (angle per unit interval) of direction X towards Y as we go along Z. Thus C^Y_{XX} is the line curvature of coordinate line X in the surface direction X towards Y. For circular (space-plane) rotation, when X turns towards Y, Y turns *away* from X, but for hyperbolic (distance-time) rotation, when X turns towards T, T turns *towards* X.

Hence $C^Y_{XX} = -C^X_{YX}$ ....(4*a*), but $C^T_{XX} = +C^X_{TX}$.........(4*b*)

C^Y_{XXY}, or C_{XY}, is the inherent curvature of the coordinate surface XY, that is, the net rotation of the XY orientation as we go round a circuit enclosing unit area, travelling first in the positive X direction and turning into the positive Y direction. If we traverse a circuit in the reverse sense, the rotation will be reversed. Hence

$$C_{XY} = C^Y_{XXY} = -C^Y_{XYX} = +C^X_{YYX} = C_{YX}, \quad \text{............(5a)}$$

but

$$C_{XT} = C^T_{XXT} = -C^T_{XTX} = -C^X_{TTX} = -C_{TX}. \quad \text{..........(5b)}$$

AUXILIARY NOMENCLATURE FOR CURVATURE.

The following is a summary of terms used in connection with curvature:

Relative Curvature of an element (line, surface or solid) is the curvature relative to another element (that is, the rate at which one element curves away from the other).

Apparent Curvature is the curvature relative to what would be assumed uncurved if our fourfold were Euclidean.

True Curvature is the curvature in the fourfold as measured by means of length (area, etc.) on either side of the element.

An element whose true curvature is zero is said to be *Direct* (or *geodesic*).

***Inherent Curvature* is the curvature of a surface (solid, or fourfold) as found by measurements *in* the surface (etc.).**

***Residual Curvature* is the inherent curvature of a *direct* surface tangent to the surface considered.**

STEADY ACCELERATION FIELD.

Throughout this paper we shall confine ourselves to a static field, that is to say, we assume (1) that every length and every angle in our coordinate space-frame remains unaltered throughout the time (the space-frame is " rigid "); (2) that the constrained acceleration to which the frame is subject (that is, the acceleration which a particle embedded in the frame is forced to undergo relative to the path it would follow if it were free) is constant throughout the time at every point in the frame, though it may vary from point to point of the frame. From condition (1) it follows that the time-coordinate lines are everywhere normal to the space-frame and parallel to each other (that is, remain the same distance apart): hence the space-frames are direct. From (2) it follows that each time-coordinate line has uniform curvature throughout its length, but that different time-coordinate lines will in general have different curvatures.

For convenience of reference, as we have only one purpose in view in this paper, namely, the examination of the sun's field, we shall call our coordinates R, Θ, Φ, and T, and shall choose the direction of constrained acceleration at every point as the direction of the coordinate line R (that is, the coordinate lines R lie along the " lines of force " but in the opposite sense). The remaining space-coordinate directions Θ, Φ, being normal to R, lie in the equipotential surfaces.

As we have already done for time, so we shall now do for the other coordinates, namely, distinguish between coordinate and true intervals by using small letters for the coordinate intervals and capital letters for the true intervals. For the ratio of the true to the coordinate interval we shall use the letter g with an appropriate suffix. Thus

$$dR = g_r\,dr,\quad d\Theta = g_\theta\,d\theta,\quad d\Phi = g_\phi\,d\phi,\quad dT = g_t\,dt. \qquad \text{.........(6)}$$

Since R is everywhere normal to T, we have, by formula (2*f*),

$$C^R_{TT} = \frac{1}{\delta T}\cdot\frac{\partial(\delta T)}{\partial R} = \frac{1}{g_t\delta t}\cdot\frac{\partial(g_t\delta t)}{\partial R} = \frac{1}{g_t}\frac{\partial g_t}{\partial R} = \frac{1}{g_t g_r}\frac{\partial g_t}{\partial r}. \qquad \text{.......(7a)}$$

Similarly $$C^R_{\Theta\Theta} = -\frac{1}{g_\theta}\frac{\partial g_\theta}{\partial R} = -\frac{1}{g_\theta g_r}\frac{\partial g_\theta}{\partial r}, \qquad \text{..............................(7b)}$$

and $$C^R_{\Phi\Phi} = -\frac{1}{g_\phi}\frac{\partial g_\phi}{\partial R} = -\frac{1}{g_\phi g_r}\frac{\partial g_\phi}{\partial r}. \qquad \text{..............................(7c)}$$

The direct path of a particle in space-time we call S. u is the

rapidity of the particle relative to our space-frame, that is, it is the hyperbolic angle between T and S. We further define u_T^R, u_Φ^R, and u_Θ^R as follows:

$$\tanh u_T^R = \frac{\dfrac{dR}{dS}}{\dfrac{dT}{dS}} \quad \text{...(8}a\text{)}, \qquad \tan u_\Theta^R = \frac{\dfrac{dR}{dS}}{\dfrac{d\Theta}{dS}} \quad \text{...(8}b\text{)}, \qquad \tan u_\Phi^R = \frac{\dfrac{dR}{dS}}{\dfrac{d\Phi}{dS}} \quad \text{...(8}c\text{)}.$$

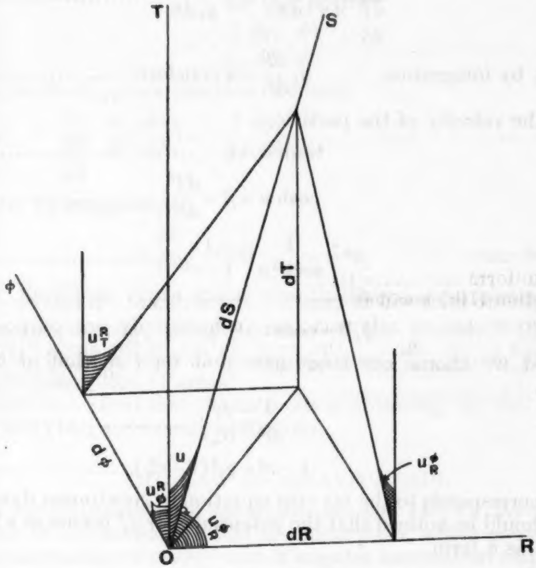


FIG. 5.

As we travel along S our coordinate orientation will rotate owing to the curvature of the coordinate lines T, Θ and Φ (also owing to twists, which, however, we shall not need to consider). The S direction, though really constant, will therefore appear to rotate by an equal amount in the opposite direction. The amount of rotation for an infinitesimal element dS is the geometrical sum of the separate amounts due to the components dT, dR, $d\Theta$, and $d\Phi$, of dS. Since the space-solid $R\Theta\Phi$ is direct, the only component which will give rotation involving T is dT, which gives rotation involving R and T. Thus we have

$$-du_T^R = C_{TT}^R \frac{dT}{dS} \, . \, dS. \quad \text{..........................(9)}$$

Any increment of dT (dS remaining constant) will arise from the

rotation of dR, and will be proportional to it. Hence

$$\frac{d}{dS}\left(\frac{dT}{dS}\right) = +\frac{dR}{dS}\frac{du_T^R}{dS} = -\frac{dR}{dS}C_{TT}^R\frac{dT}{dS} = -\frac{dR}{dS}\cdot\frac{1}{g_t}\frac{\partial g_t}{\partial R}\cdot\frac{dT}{dS}.$$

Since g_t varies only along R, this is equal to

$$-\frac{1}{g_t}\frac{dg_t}{dS}\cdot\frac{dT}{dS}. \quad \text{.........................(10a)}$$

Thus
$$\frac{1}{\frac{dT}{dS}}\cdot\frac{d}{dS}\left(\frac{dT}{dS}\right) = -\frac{1}{g_t}\frac{dg_t}{dS}, \quad \text{.........................(10b)}$$

whence, by integration, $\quad g_t\frac{dT}{dS} = \text{a constant.}$ ....(10c)

If v is the velocity of the particle,

$$\tanh u = v, \quad \text{.........................(11a)}$$

and
$$\cosh u = \beta = \frac{dT}{dS}, \quad \text{.........................(11b)}$$

whence
$$\beta^2 = \frac{1}{\operatorname{sech}^2 u} = \frac{1}{1-v^2}. \quad \text{.........................(11c)}$$

Equation (10c) becomes

$$g_t\beta = \text{const.} = \beta_\infty, \quad \text{.........................(12a)}$$

provided we choose our coordinate unit for t so that at infinity $dt = dT$.

Thus
$$\frac{1}{\beta^2} = \frac{g_t^2}{\beta_\infty^2}; \quad \text{.........................(12b)}$$

that is,
$$1 - v^2 = g_t^2(1 - v_\infty^2). \quad \text{.................(12c)}$$

This corresponds to the *vis viva* equation in Newtonian dynamics, and it should be noticed that the potential g_t or g_t^2 occurs as a factor, and not as a term.

CENTRAL FORCE.

When the constrained acceleration to which our reference frame is subject radiates from a fixed centre O, and is a function of the distance from O, our coordinate lines R become directs radiating from O. If we then take ϕ as the longitude and θ as the latitude, the coordinate curvatures C_T, C_R, C_Θ, are all functions of r only. Hence g_t, g_r, g_θ are also functions of r only. g_ϕ is a function of r multiplied by $\cos\theta$.

We can choose θ so that initially the particle is moving in the plane $\theta = 0$. In this case there is no rotation involving Θ, and the direct path of the particle will therefore lie entirely in the threefold $\theta = 0$. We therefore have

$$-du_\Phi^R = C_{\Phi\Phi}^R\frac{d\Phi}{dS}.dS = -\frac{1}{g_\phi}\frac{\partial g_\phi}{\partial R}d\Phi. \quad \text{.................(13)}$$

Thus $$\frac{d}{dS}\left(\frac{d\Phi}{dS}\right)=\frac{dR}{dS}\cdot\frac{du_R^\Phi}{dS},$$ there being no other rotation involving Φ.

$$=-\frac{dR}{dS}\cdot\frac{du_\Phi^R}{dS}$$

$$=\frac{dR}{dS}\cdot C_{\Phi\Phi}^R\frac{d\Phi}{dS}$$

$$=-\frac{dR}{dS}\cdot\frac{1}{g_\phi}\frac{\partial g_\phi}{\partial R}\cdot\frac{d\Phi}{dS}$$

$$=-\frac{1}{g_\phi}\frac{dg_\phi}{dS}\cdot\frac{d\Phi}{dS}, \quad\text{.............................}(14a)$$

since, for $\theta=0$, g_ϕ is a function of r only.

Hence $$\frac{1}{\frac{d\Phi}{dS}}\cdot\frac{d}{dS}\left(\frac{d\Phi}{dS}\right)=-\frac{1}{g_\phi}\frac{dg_\phi}{dS}, \quad\text{.................................}(14b)$$

whence, by integration,

$$g_\phi\frac{d\Phi}{dS}=\text{const.} \quad\text{.......................................}(14c)$$

For Euclidean space (polar coordinates) $g_\phi=r$; also $\frac{d\Phi}{dS}$ is the transverse velocity, using the time of the particle. Hence the formula (14*c*) corresponds to the equation $r^2\dot\phi=pv=h$, giving the equable description of areas.

It will be noticed that the methods of obtaining the two formulae (10*c*) and (14*c*) are exactly analogous.

THE LAW OF GRAVITATION.

We have obtained two equations (10*c* and 14*c*) for the path of a particle in a symmetrical field of acceleration, which correspond to the conservation of energy and of angular momentum respectively. In order to use these equations to determine the path completely we shall need to know the values of the g's at every point of the field. In space-time which is homaloidal (quasi-euclidean) the g's are fixed by the choice of a coordinate system, but since the space-time we are using is non-euclidean, we cannot use the properties of quasi-euclidean geometry to relate the values of the g's in different parts of the field. We have to find, if we can, a differential law limiting the variations of the g's from point to point of the field. We have already seen that g_t corresponds (or at least is analogous) to the potential in Newtonian theory, and we may therefore expect to obtain our law as a modification or generalisation of Laplace's Equation for the variation of the potential in the gravitational field.

The field differs from a quasi-euclidean space-time in that its surfaces have residual curvature, so that direct lines which start parallel do not remain parallel, but converge or diverge. We therefore proceed to find expressions for the residual curvatures in terms

of the g's. We have (see Fig. 2):

$$C_{RT} = C^T_{RRT} = \frac{\text{angle discrepancy}^T_R}{\text{area}} \text{ for surface-element } dr\,.\,dt$$

$$= \frac{\text{rotation}^T_R \text{ for circuit } (dr + dt - dr - dt)}{g_r dr\,.\,g_t dt}$$

$$= \frac{\frac{\partial}{\partial r}(C^T_{RT}\,.\,g_t dt)\,dr}{g_r g_t\,dr\,dt} \quad \text{(There is no rotation for the elements } dr \text{ and } -dr,\ R \text{ being direct.)}$$

$$= \frac{1}{g_r g_t}\frac{\partial}{\partial r}(C^R_{TT}\,g_t), \text{ using (4}b\text{)}$$

$$= \frac{1}{g_t}\frac{\partial^2 g_t}{\partial R^2}, \text{ using (6) and (7}a\text{).} \qquad \text{(15}a\text{)}$$

Similarly
$$C_{R\Theta} = \frac{\frac{\partial}{\partial r}(C^\Theta_{R\Theta}\,.\,g_\theta d\theta)\,dr}{g_r g_\theta\,dr\,d\theta}$$

$$= \frac{1}{g_r g_\theta}\frac{\partial}{\partial r}(-C^R_{\Theta\Theta}\,.\,g_\theta), \text{ using (4}a\text{)}$$

$$= \frac{1}{g_\theta}\frac{\partial^2 g_\theta}{\partial R^2}, \text{ using (7}b\text{).} \qquad \text{(15}b\text{)}$$

Since each R-line is direct, and the curvature of T is entirely in the direction of R, that is, in the TR surface, the TR surface is direct in the fourfold. This applies also to $R\Theta$, and to $R\Phi$ when $\theta = 0$. Hence the inherent curvatures C_{RT} and $C_{R\Theta}$ are entirely residual, as also $C_{R\Phi}$ when $\Theta = 0$.

Since the TR surface is direct, a time-line S, which is tangent to T at $t = 0$, and which is direct in the surface TR, is also direct in the fourfold. For the whole of a TR surface, and therefore for each S-direct, θ and ϕ are constant. Consider the instantaneous circle (or rather meridian) ϕ = constant, r = constant, $t = 0$. The S-directs tangent to T through points on this circle will generate a surface ΘS, which has true curvature only along Θ, and whose inherent curvature is therefore entirely residual. Also Θ, being normal to TR, is everywhere normal to S. We therefore have

$$\bar{C}_{S\Theta} = C^\Theta_{SS\Theta} = \frac{\text{angle discrepancy}^\Theta_S}{\text{area}} \text{ for surface element } ds\,.\,d\theta$$

$$= \frac{\text{rotation}^\Theta_S \text{ for circuit } (ds + d\theta - ds - d\theta)}{g_s ds\,.\,g_\theta d\theta}$$

$$= \frac{\frac{\partial}{\partial s}(C^\Theta_{S\Theta}\,.\,g_\theta d\theta)\,.\,ds}{g_\theta g_s\,.\,d\theta\,ds} \quad \text{(there is no rotation for } ds \text{ and } -ds\text{)}$$

$$= \frac{1}{g_\theta g_s}\frac{\partial}{\partial s}(C^S_{\Theta\Theta}\,g_\theta), \text{ using (4}b\text{)}$$

$$= \frac{1}{g_\theta}\frac{\partial}{\partial S}\left(\frac{\partial g_\theta}{\partial S}\right) \qquad \text{(15}c\text{)}$$

Since $S\Theta$ is tangent to $T\Theta$ at $t=0$, the value of $\bar{C}_{S\Theta}$ at $t=0$ is also the residual curvature of $T\Theta$.

In the Newtonian theory no distinction is made between what we have called the " true time " and the " coordinate time ", but it will appear that where the time refers to measurements made *at* a place, the true time must be taken, while, when we are comparing instants at different places of our assumed rigid space, we must use the coordinate time. Hence to compare the accelerations at different places, we must measure the acceleration as the rate of increase of *true velocity* in *coordinate time*, that is, by C^R_{Tt} instead of C^R_{TT}. It will be remembered that C^R_{Tt} is the quantity which, in the case of constant acceleration which we first considered, we found to be uniform all over the field. For the coordinates, we are using

$$C^R_{Tt}=C^R_{TT}\cdot\frac{dT}{dt}=g_tC^R_{TT}=\frac{\partial g_t}{\partial R}. \quad \text{.................}(16)$$

If P is the potential in the Newtonian theory, then Laplace's Equation gives

$$\frac{\partial^2 P}{\partial X^2}+\frac{\partial^2 P}{\partial Y^2}+\frac{\partial^2 P}{\partial Z^2}=0, \quad \text{....................}(17a)$$

where capital letters are used because X, Y, Z are to be the *true* distances.

Now $\dfrac{\partial^2 P}{\partial X^2}=\dfrac{\partial}{\partial X}\displaystyle\int^X_{Tt}$ where $\displaystyle\int^X_{Tt}$ is the rate of increase of true X-velocity in coordinate time,

$$=\frac{\partial}{\partial X}(-C^X_{Tt}) \text{ where } C^X_{TT} \text{ is the true curvature of coordinate line } T,$$

$$=-\frac{\partial}{\partial X}\left(\frac{\partial g_t}{\partial X}\right)$$

$$=g_tC_{TX}, \text{ provided } X \text{ is direct. } \quad \text{........................}(17b)$$

In the ordinary application of Laplace's Equation, X, Y, Z are assumed to be straight, and their direction to be unaltered with the time. In our present application we must therefore take X, Y, Z to be direct, and the " twists " C^Y_{XT}, C^Z_{YT}, C^X_{ZT}, to be all zero. This means that the surfaces TX, TY, TZ, can only be curved in one direction ; hence their inherent curvature must be entirely residual.

The factor g_t occurs in all three terms. Hence Laplace's Equation is equivalent to

$$\bar{C}_{TX}+\bar{C}_{TY}+\bar{C}_{TZ}=0, \quad \text{........................}(17c)$$

where the bars indicate residual curvature.

Since Laplace's Equation takes account only of the accelerations *at the same moment* at neighbouring points of the field, it is unaffected by changes occurring in the field. By the Principle of Relativity we should therefore expect it to hold for every possible space and the appropriate time (the space must, of course, not be curved

in the fourfold in the infinitesimal region to which we apply the equation). Hence whatever time-direction we choose, the sum of the residual curvatures of any three mutually normal surfaces containing the time-direction will be zero. For this to be possible, we must have

$$C_{XT}+C_{XY}+C_{XZ}=0,\quad C_{YT}+C_{YX}+C_{YZ}=0,$$
$$C_{ZT}+C_{ZX}+C_{ZY}=0. \quad \text{....(17}d\text{)}$$

Our differential law of gravitation will therefore include equations (17*c*) and (17*d*).

Another interpretation of Laplace's Equation may be found as follows:

$$\frac{\partial^2 P}{\partial X^2}=\frac{\partial}{\partial X}\int_{Tt}^{X}=\frac{\partial}{\partial X}\frac{\partial}{\partial t}(v_T^X), \quad \text{......................................(18}a\text{)}$$

where v_T^X is the X-component of the velocity of a particle moving freely in the field,

$$=\frac{1}{\delta X}\cdot\delta_X\left(\frac{\partial}{\partial t}v_T^X\right)=\frac{1}{\delta X}\cdot\frac{\partial}{\partial t}(\delta_X v_T^X)=\frac{1}{\delta X}\frac{\partial}{\partial t}\frac{\partial}{\partial T}(\delta X). \quad \text{.....(18}b\text{)}$$

Consider a set of eight particles moving freely in the field and initially at rest at the corners of a rectangular solid element δX, δY, δZ. If V be the volume of the solid element determined by the particles at any moment,

$$V=\delta X\,.\,\delta Y\,.\,\delta Z\,; \quad \text{..(19}a\text{)}$$

$$\frac{dV}{dT}=\sum\frac{d}{dT}(\delta X)\,.\,\delta Y\,.\,\delta Z\,; \quad \text{......................................(19}b\text{)}$$

$$\frac{d}{dt}\frac{dV}{dT}=\sum\frac{d}{dt}\frac{d}{dT}(\delta X)\,.\,\delta Y\,.\,\delta Z+\sum\frac{d}{dT}(\delta X)\,.\,\frac{d}{dt}(\delta Y)\,.\,\delta Z. \quad \text{...(19}c\text{)}$$

Since initially $\frac{dX}{dT}=\frac{dY}{dT}=\frac{dZ}{dT}=0$ for all the particles, the second term vanishes, and we have

$$\frac{1}{V}\frac{d}{dt}\frac{dV}{dT}=\sum\frac{1}{\delta X}\cdot\frac{d}{dt}\frac{d}{dT}(\delta X)=\sum\frac{\partial^2 P}{\partial X^2}. \quad \text{...........(19}d\text{)}$$

Hence Laplace's Equation means that the second time-flux of the space-volume determined by free particles momentarily at rest is zero. This can be extended to a finite space-volume, provided our space-solids are all direct (which means that our space is rigid, as already assumed), and that the second differentiation is with respect to t instead of T.

THE SUN'S FIELD.

With the coordinates we have already adopted for the sun's field, our law of gravitation gives

$$\left.\begin{aligned}\bar{C}_{TR}+\bar{C}_{T\Theta}+\bar{C}_{T\Phi}&=0\\ \bar{C}_{RT}+\bar{C}_{R\Theta}+\bar{C}_{R\Phi}&=0\\ \bar{C}_{\Theta T}+\bar{C}_{\Theta R}+\bar{C}_{\Theta\Phi}&=0\\ \bar{C}_{\Phi T}+\bar{C}_{\Phi R}+\bar{C}_{\Phi\Theta}&=0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

By symmetry, $\bar{C}_{R\Theta}=\bar{C}_{R\Phi}$ and $\bar{C}_{T\Theta}=\bar{C}_{T\Phi}$. Also by (5), $\bar{C}_{R\Theta}=\bar{C}_{\Theta R}$ and $\bar{C}_{RT}=-\bar{C}_{TR}$. It is now easy, by addition and subtraction of the four equations (20), to show that

$$\bar{C}_{R\Theta}=\bar{C}_{R\Phi}=\bar{C}_{\Theta T}=\bar{C}_{\Phi T}=\bar{C} \text{ (say)}; \quad \ldots\ldots\ldots\ldots(21a)$$

and

$$\bar{C}_{RT}=\bar{C}_{\Theta\Phi}=-2\bar{C}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(21b)$$

The g's are functions of r only, except for the factor $\cos\theta$ in $g_\phi=g_\theta\cos\theta$. We may, therefore, without altering the properties of the field, use a function of r instead of r, and we accordingly use g_θ, so as to make

$$d\Theta=r\,d\theta, \quad d\Phi=r\cos\theta\,.\,d\phi, \quad \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

as in ordinary polar coordinates.

Consider the volume contained between the spherical surfaces $r=r_1$, $r=r_2$, at time $t=0$. Let a variable volume V be determined by free particles which start from rest on these bounding surfaces at $t=0$. Then

$$\frac{d}{dt}\left(\frac{dV}{dT}\right)=\left[4\pi r^2\frac{d}{dt}\left(\frac{dR}{dT}\right)\right]_{r_1}^{r_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(23a)$$

By (17a) and (19d) (our second interpretation of Laplace), the first member of equation (23a) is equal to zero. Thus

$$4\pi r^2\cdot\frac{d}{dt}\left(\frac{dR}{dT}\right) \text{ has the same value for } r=r_1 \text{ as for } r=r_2.$$

It must therefore be independent of r, so that

$$4\pi r^2\frac{d}{dt}\left(\frac{dR}{dT}\right)=\text{const.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(23b)$$

Hence for a free particle,

$$\frac{d}{dt}\left(\frac{dR}{dT}\right)=\frac{\text{const.}}{r^2}=-\frac{m}{r^2}, \quad \ldots\ldots\ldots\ldots(23c)$$

where the constant m corresponds to the gravitational mass of the sun in the Newtonian theory.

The four residual curvatures $\bar{C}_{R\Theta}$, $\bar{C}_{\Theta\Phi}$, $\bar{C}_{\Theta T}$, $\bar{C}_{RT}$ may all be calculated independently in terms of r and the g's, and by using the relations (21), we can then obtain the g's as functions of r.

From (15a),

$$\bar{C}_{RT}=\frac{1}{g_t}\frac{\partial^2 g_t}{\partial R^2}=\frac{1}{g_t}\frac{\partial}{\partial R}C^R_{Tt}=\frac{1}{g_tg_r}\frac{\partial}{\partial r}\left(\frac{m}{r^2}\right)=-\frac{1}{g_tg_r}\frac{2m}{r^3}. \quad \ldots\ldots\ldots(24)$$

From (15b),

$$\bar{C}_{R\Theta}=\frac{1}{g_\theta}\frac{\partial^2 g_\theta}{\partial R^2}=\frac{1}{r}\frac{\partial}{g_r\partial r}\left(\frac{\partial r}{g_r\partial_r}\right)=\frac{1}{rg_r}\cdot\frac{\partial}{\partial r}\left(\frac{1}{g_r}\right)=-\frac{1}{rg_r{}^3}\frac{\partial g_r}{\partial r}. \quad \ldots\ldots(25)$$

The residual curvature of $T\Theta$ is that of the tangent surface $S\Theta$ for which formula (15*c*) was obtained. Hence

$$\bar{C}_{T\Theta}=C_{S\Theta}=\frac{1}{g_\theta}\frac{\partial^2 g_\theta}{\partial S^2}=\frac{1}{r}\frac{\partial^2 r}{\partial S^2}=\frac{1}{r}\frac{\partial}{\partial S}\left(\frac{dr}{dT}\frac{dT}{dS}\right)=\frac{1}{r}\frac{d^2 r}{dT^2},$$

since, at $t=0$, $\dfrac{dT}{dS}$ is stationary and equal to 1;

$$=\frac{1}{r}\frac{d}{dT}\left(\frac{dR}{g_r\,dT}\right)=\frac{1}{rg_r}\frac{d^2R}{dT^2},\ \text{since } g_r \text{ is stationary};$$

$$=-\frac{1}{rg_r}C^R_{TT}=-\frac{1}{rg_r}\cdot\frac{1}{g_t}\frac{\partial g_t}{\partial R}=-\frac{1}{rg_r{}^2g_t}\frac{\partial g_t}{\partial r},$$

and so $\bar{C}_{\Theta T}=-\bar{C}_{T\Theta}=\dfrac{1}{rg_r{}^2g_t}\dfrac{\partial g_t}{\partial r}$. ....(26)

The total curvature (that is, the total angle discrepancy) of the surface $\Theta\Phi$ is -4π. Therefore its inherent curvature (which is uniform)

$$=\frac{\text{angle discrepancy}}{\text{total area}}=\frac{-4\pi}{4\pi r^2}=-\frac{1}{r^2}. \quad \text{..........................(27a)}$$

The true curvature of $\Theta\Phi=-C^R_{\Theta\Theta}\,.\,C^R_{\Phi\Phi}$, where $C^R_{\Phi\Phi}$ is taken along $\theta=0$;

$$=-\left(\frac{1}{g_\theta}\frac{\partial g_\theta}{\partial R}\right)^2=-\frac{1}{r^2}\left(\frac{\partial r}{g_r\partial r}\right)^2=-\frac{1}{r^2g_r{}^2}. \quad \text{..................(27b)}$$

Thus the residual curvature

$$\bar{C}_{\Theta\Phi}=-\left(\frac{1}{r^2}-\frac{1}{r^2g_r{}^2}\right)=\frac{1}{r^2}(1-g_r{}^2). \quad \text{..............................(27c)}$$

We have now considerable choice in the use of equations (24) to (27) to find g_t, g_r, in terms of r. The following is one simple method.

From (21*a*), we have $\quad \bar{C}_{\Theta T}-\bar{C}_{R\Theta}=0.$

Using the values found,

$$\frac{1}{rg_r{}^2g_t}\frac{\partial g_t}{\partial r}+\frac{1}{rg_r{}^3}\frac{\partial g_r}{\partial r}=0. \quad \text{..........................(28a)}$$

Multiplying by $rg_r{}^3g_t$ and integrating, we obtain $g_tg_r=\text{constant}$. If we assume that at infinity $g_t=1=g_r$, we have

$$g_tg_r=1. \quad \text{..........................(28b)}$$

Now $$C^R_{Tt}=\frac{\partial g_t}{\partial R}=\frac{g_t\partial g_t}{g_tg_r\partial r}=\tfrac{1}{2}\frac{\partial g_t{}^2}{\partial r}. \quad \text{....................(29a)}$$

But from (23*c*) $\quad C^R_{Tt}=\dfrac{m}{r^2}$; $\quad$ hence $\quad \dfrac{\partial g_t{}^2}{\partial r}=\dfrac{2m}{r^2}$. ....(29b)

At infinity $\quad g_t=1$; $\quad$ thus $\quad g_t{}^2=1-\dfrac{2m}{r}$. ....(29c)

Hence $$g_t=\frac{1}{g_r}=\sqrt{\left(1-\frac{2m}{r}\right)}. \quad \text{..........(29d)}$$

We can now find the true curvatures of the coordinate lines in terms of r. We have already assumed (from symmetry) that R is direct, and that T and Θ are curved only towards R (or $-R$), the same applying to Φ so long as we confine ourselves to the plane $\theta=0$.

$$C^R_{TT}=\frac{1}{g_t}\frac{\partial g_t}{\partial R}=\frac{1}{g_t g_r}\frac{\partial g_t}{\partial r}=\frac{\partial g_t}{\partial r}=\frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-\frac{1}{2}}$$
$$=\frac{m}{r^2}+\frac{m^2}{r^3}+\dots, \qquad (30a)$$

$$C^R_{\Theta\Theta}=-\frac{1}{g_\theta}\frac{\partial g_\theta}{\partial R}=-\frac{1}{r}\frac{\partial r}{g_r\partial r}=-\frac{1}{rg_r}=-\frac{1}{r}\left(1-\frac{2m}{r}\right)^{\frac{1}{2}}$$
$$=-\frac{1}{r}+\frac{m}{r^2}+\frac{1}{2}\frac{m^2}{r^3}+\dots \qquad (30b)$$

A direct tangent to T will have apparent curvature

$$\bar{C}_T=-\frac{m}{r^2}-\frac{m^2}{r^3}-\dots, \qquad (30c)$$

representing the ordinary gravitational effect modified by a slight correction. Since in Euclidean space we should assume Θ to have curvature $-\frac{1}{r}$, a direct tangent to Θ (or to Φ at $\Theta=0$) will have apparent curvature

$$\bar{C}_\Theta=-\frac{m}{r^2}-\frac{1}{2}\frac{m^2}{r^3}-\dots \qquad (30d)$$

It will be noticed that these apparent curvatures of the space and time directs, though sensibly equal for all ordinary values of r, differ in the coefficients after the first term.

THE SPECTRAL SHIFT.

We have already found (formula (3)) the non-correspondence of true time-intervals at different parts of the field in the case of uniform acceleration. This effect depends on the value of g_t, which varies from point to point of the field. In the sun's field

$$g_t=\sqrt{\left(1-\frac{2m}{r}\right)}=1-\frac{m}{r}$$

approximately. Since all the solids $t=$constant are direct and normal to T, contemporaneous events will be those for which t, and not T, has the same value. Hence a coordinate time-interval dt will appear at $r=r_1$ as a true interval of $g_t\,dt=\left(1-\frac{m}{r_1}\right)dt$, and at $r=r_2$ as a true interval of $\left(1-\frac{m}{r_2}\right)dt$. Hence, if ϖ be the period, and λ the wave-length, of light emitted by an atom, at $r=r_1$, the period of the light received at $r=r_2$ will be

$$\varpi' = \varpi \cdot \frac{1-\frac{m}{r_2}}{1-\frac{m}{r_1}} = \varpi\left(1-\frac{m}{r_2}+\frac{m}{r_1}\right) \text{ approximately. } \ldots\ldots(31a)$$

The period is therefore lengthened by the amount $\varpi\left(\frac{m}{r_1}-\frac{m}{r_2}\right)$ and the wavelength $\lambda = \varpi c$ in the same proportion.

The spectral shift may also be obtained in the following manner. The space through which the light is travelling is free, and therefore does not share in the constrained acceleration to which our frame of reference is subject. It has therefore, relative to that frame, an acceleration of $f^R_{Tt} = \frac{m}{r^2}$ towards the sun. Hence, by the time the light has traversed the distance from the sun to the earth, this space (which was the sun's space at the moment the light was emitted) has attained a velocity relative to our frame, and therefore relative to the earth (we should naturally allow for the earth's orbital motion) of

$$\int_{r=r_1}^{r=r_2} \frac{m}{r^2}\,dt = \int_{r_1}^{r_2} \frac{m}{r^2}\frac{dt}{dr}\,dr = \int_{r_1}^{r_2} \frac{1}{c}\frac{m}{r^2}\,dr$$

approximately, where c is the velocity of light

$$= \frac{1}{c}\left(\frac{m}{r_1}-\frac{m}{r_2}\right). \quad \ldots\ldots\ldots\ldots(31b)$$

Hence by the Doppler Effect the period and wavelength will be lengthened in the proportion of $\left(1+\frac{m}{r_1}-\frac{m}{r_2}\right) : 1$.

DEFLECTION OF LIGHT IN THE SUN'S FIELD.

The path of a light-pulse in space-time is direct, but will have apparent curvature relative to our accelerated frame. Choose the coordinate space-plane $\theta = 0$, so that it contains the initial space-path of the light. Since both plane and path are direct, the path

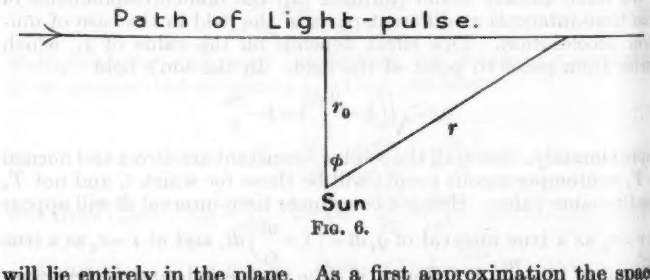


FIG. 6.

will lie entirely in the plane. As a first approximation the space-path is straight, and follows the line $\theta = 0$, $r\cos\phi = r_0$, $r\sin\phi = t$, where r_0 is the value of r at the nearest approach of the light to the sun. As we travel along the path of the light-pulse the space-

orientation of our coordinate system rotates at the rate

$$C^R_{\Phi\Phi}\, d\Phi = \left(-\frac{1}{r} + \frac{m}{r^2}\right) d\Phi$$

very nearly. Euclidean geometry would lead us to expect a rotation of $-\frac{1}{r}\, d\Phi$. Allowing for this we have a net rotation of $\frac{m}{r^2}\, d\Phi$, which gives rise to an apparent deflection of the ray of light, due to space distortion, of

$$-\frac{m}{r^2}\, d\Phi = -\frac{m}{r}\, d\phi = -\frac{m}{r_0} \cos\phi \,.\, d\phi. \quad \text{.........(32a)}$$

Meanwhile the space through which the light is travelling has an acceleration towards the sun of $-C^R_{Tt} = \frac{m}{r^2}$. One component of this acceleration acts along the path and so produces no deflection, but only a change of frequency of the light, as already explained. The component which acts normal to the path produces a normal velocity of

$$\begin{aligned} -C^R_{TT}\, dT \,.\, \cos\phi = -\frac{m}{r^2}\, dt \,.\, \cos\phi &= -\frac{m}{r_0^2} \cos^2\phi\, d(r \sin\phi) \cos\phi \\ &= -\frac{m}{r_0^2} \cos^3\phi\, d(r_0 \tan\phi) \\ &= -\frac{m}{r_0^2} \cos^3\phi \,.\, \sec^2\phi \,.\, r_0\, d\phi \\ &= -\frac{m}{r_0} \cos\phi \,.\, d\phi. \quad \text{............(32b)} \end{aligned}$$

Since the velocity of light is unity, the rate of production of normal velocity is also the measure of the deflection produced. The combined deflection of the space-path produced by the space and time curvatures is therefore

$$-\frac{2m}{r_0} \cos\phi \,.\, d\phi, \quad \text{........................(32c)}$$

and the total deflection of a ray travelling from minus infinity to plus infinity is

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{2m}{r_0} \cos\phi \,.\, d\phi = \frac{4m}{r_0}. \quad \text{....................(32d)}$$

PATH OF A PLANET.

Let the planet begin to move in the plane $\theta = 0$. Since the path is direct, the motion will be entirely confined to this plane. For *any* time-path in the field

$$dR^2 + d\Theta^2 + d\Phi^2 - dT^2 = -dS^2. \quad \text{..........(33a)}$$

Hence

$$\left(\frac{dR}{d\Phi}\frac{d\Phi}{dS}\right)^2 + 0 + \left(\frac{d\Phi}{dS}\right)^2 - \left(\frac{dT}{dS}\right)^2 = -1. \quad \text{............(33b)}$$

By (10*c*) and (12*a*), $\dfrac{dT}{dS}=\dfrac{\beta_\infty}{g_t}$.

By (14*c*), $\dfrac{d\Phi}{dS}=\dfrac{h}{g_\phi}=\dfrac{h}{r}$.

Substituting in (33*b*), we have

$$\left(\frac{g_r}{r}\frac{dr}{d\phi}\cdot\frac{h}{r}\right)^2+\left(\frac{h}{r}\right)^2-\frac{\beta_\infty^2}{g_t^2}+1=0. \quad \text{.................(33c)}$$

Multiplying by $\dfrac{g_t^2}{h^2}$, and using (28*b*) and (29*c*),

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2+\frac{1}{r^2}\left(1-\frac{2m}{r}\right)-\frac{\beta_\infty^2}{h^2}+\frac{1}{h^2}\left(1-\frac{2m}{r}\right)=0. \quad \text{.................(33d)}$$

Put $\dfrac{1}{r}=u$, so that $-\dfrac{1}{r^2}\dfrac{dr}{d\phi}=\dfrac{du}{d\phi}$.

Then $$\left(\frac{du}{d\phi}\right)^2+u^2-2mu^3-\frac{2m}{h^2}u+\frac{1-\beta_\infty^2}{h^2}=0. \quad \text{.................(33e)}$$

Differentiating and dividing by $2\dfrac{du}{d\phi}$,

$$\frac{d^2u}{d\phi^2}+u-\frac{m}{h^2}=3mu^2. \quad \text{..............................(34a)}$$

Put $$u-\frac{m}{h^2}=u-\frac{1}{l}=u-a=v.$$

Equation (34*a*) then becomes

$$\frac{d^2v}{d\phi^2}+v=3m(v^2+2va+a^2), \quad \text{...............(34b)}$$

or $$\frac{d^2v}{d\phi^2}+v(1-6ma)=3ma^2+3mv^2. \quad \text{......................(34c)}$$

Ignoring $3mv^2$, we have as a first approximation

$$v=3ma^2+ae\cos\sqrt{1-6ma}\,(\phi-\varpi), \quad \text{..............(35a)}$$

where ϖ is the longitude of perihelion at $t=0$, a the reciprocal of the semi-latus rectum l, and e the eccentricity, of the Newtonian orbit.

Thus $$\left.\begin{aligned} v&=a(3ma+e\cos\psi),\\ \text{where}\quad \psi&=(1-3ma)(\phi-\varpi)\ \text{very nearly.}\end{aligned}\right\} \quad \text{...............(35b)}$$

The period of oscillation of v (that is, of r) bears to the period of revolution the ratio $\dfrac{1}{1-3ma}=1+3ma$ very nearly, so that the perihelion advances each revolution by a fraction of a revolution

$$3ma=\frac{3m^2}{h^2}.$$

R. A. M. K.

SOME DIFFICULT SARACENIC DESIGNS.

By E. Hanbury Hankin.

II.

A Pattern containing Seven-rayed Stars.

The ideal aimed at in Saracenic designs is that each "pattern space" should have either radial or bilateral symmetry. It is surprising that this ideal is so nearly achieved in a pattern in which the very awkward constituent of seven-rayed stars is combined with octagons and regular square-shaped outlines. This pattern is shown in Fig. 1. At first glance each of the pattern spaces seems tc be

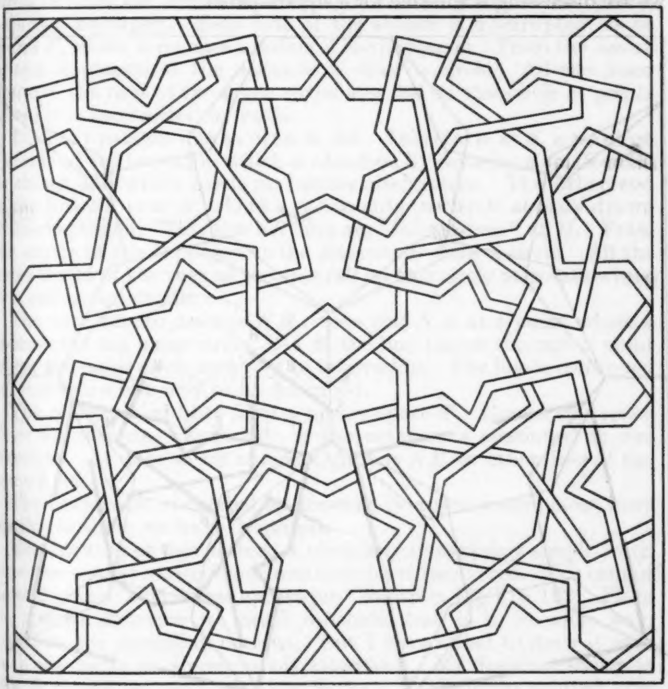


Fig. 1.

perfectly symmetrical. But on examining the trilobed spaces of which one may be seen on each side of the central point, it is easy to see that the angle between each two adjacent lobes is in two instances a right angle, but the third of these angles is a little larger than a right angle.

The illustration shows one " repeat " of this pattern. As the illustration shows four square outlines each containing an octagon, it might be thought that what is drawn is four repeats. This is not the case for the following reason. At the centre of the illustration is a diamond-shaped space. A quarter of a similar pattern space is to be found at each of the four corners. The diamond occurs, in each case, with its long axis horizontal. At the centre of each of the sides of the illustration the diamond outline occurs again, but with the long axis vertical and therefore to that extent different from the similar outlines occurring at the four corners.

A rule that is strictly followed in Saracenic art is that every panel to be decorated carries either one repeat or a whole number of repeats of its pattern. Consequently, as may be seen, the pattern we are discussing is suitable for a square panel.

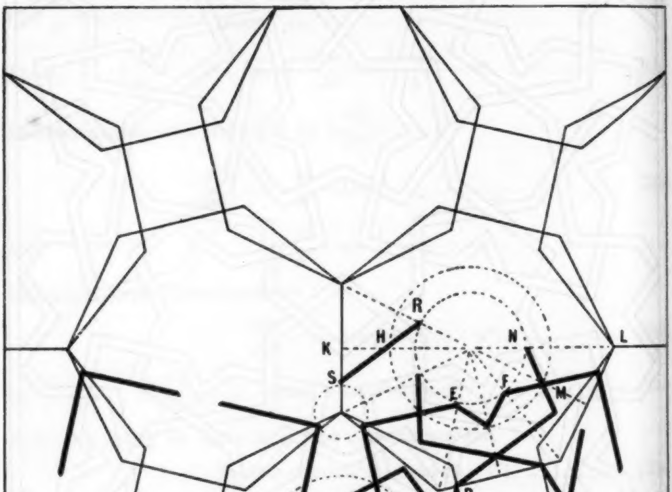


FIG. 2.

How is it to be drawn ? Suppose we have a square panel, how do we set to work to draw this pattern on it ? The method is shown in

Fig. 2. One has to begin by dividing the base AD into three parts, of which the central part BC is equal to the length of the side of a heptagon whose diameter is equal to the other two parts AB and CD. Lacking any mathematical knowledge, it was necessary for me to find this by trial and error. Having found it, and transferred it on to tracing paper, the next stage is to use it in covering the panel with a lattice of overlapping heptagons as shown in the figure. These form the chief construction lines for the pattern. Having drawn these heptagons and such of their diameters as may be necessary, one can begin drawing the pattern.

It is advisable to commence with the square outline. Its position in relation to the construction lines is shown in the figure and needs no description. The length of the side of the square is equal to the distance from the centre of a heptagon to the centre of the nearest adjacent heptagon. Each side of the square is interrupted, as at E and F, where it reaches a radius of the heptagon. From the centre of this heptagon at the distance E draw a circle. All the inner limits of the rays of the seven-rayed star fall on this circle at points where it is intersected by radii.

The next pattern line to draw is SR. This starts at S, a point on the side of the heptagon which is obtained by drawing a small circle as shown and which needs no further description. The other end of the line SR is at R, where a radius cuts the circle already drawn in the heptagon. This line SR cuts an "interradius" at H. From the centre of the heptagon at the distance H draw a circle. All the outer limits of the rays of the star fall on this circle at points where it is cut by interradii.

The next line to draw is NM. One end N is at a point where a radius cuts the inner circle. At M the line passes through a point where the outer circle is cut by an interradius. The line is continued till it meets a line now to be described.

The construction lines form a square space T. Round this space draw an octagon. Each side of the octagon is continued in one direction. It does so till it meets the line NM or other lines of the same kind.

The pattern is completed by drawing, in appropriate places, lines similar to those we have described.

The drawing of this pattern is obviously remarkably simple when once the correct construction lines have been discovered. This cannot be said of the "curvilinear arabesque" shown in Fig. 3 (p. 168). From its general structure we must conclude that it is, in some way, based on the preceding pattern. But I have failed to draw it with such regularity as is presumably attainable. My drawing, shown in Fig. 3, shows no very obvious faults. Each seven-pointed star appears fairly symmetrical. The star is surrounded by seven shield-shaped pattern spaces. These shields ought to be of the same size and symmetrical. In this I have not quite succeeded. The pattern is to be found on Plate III of Bourgoin's *Elements de l'art Arabe : Le Trait des Entrelacs* (Firmin-Ditot et Cie, Paris, 1879). But his drawing is on so small a scale that it is difficult to know how far he

has achieved regularity. Neither does it serve to give any clue to his method of construction.

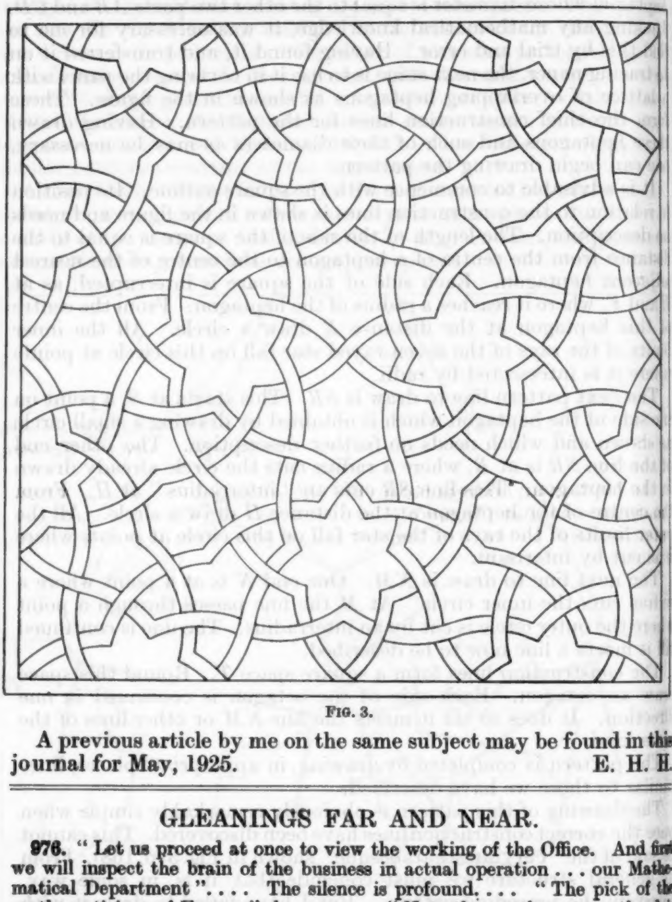


FIG. 3.

A previous article by me on the same subject may be found in this journal for May, 1925. E. H. H.

GLEANINGS FAR AND NEAR.

976. " **Let us proceed at once to view the working of the Office. And first we will inspect the brain of the business in actual operation . . . our Mathematical Department ". . . . The silence is profound. . . . " The pick of the mathematicians of Europe ", he says, . . . " Note the gentleman on the dais opposite. He is our head mathematical expert. His salary is five thousand a year. He has discovered four new kinds of infinite and can work equations in six dimensions ; mathematical psychology is his speciality, and the rapidity of his calculations is amazing. . . . Note that young girl . . . a perfect marvel, sir, a Pole, . . . and a mathematical genius. She is at work on the caprices of the Woman's Vote—one of the most puzzling speculations we have had to formulate. . . . Then the young man opposite. His subject is the Mathematical Logic of Bad Habits—a department from which we expect great moral results."— L. P. Jacks, *The Legend of Smokeover*, ch. ii. [Per Mr. P. J. Harris.]**

THE FUNCTION CONCEPT IN SCHOOL MATHEMATICS.*

BY H. R. HAMLEY.

THERE are certain topics connected with the teaching of mathematics which seem to present themselves periodically for review. The aims or objectives of mathematical teaching is one of them; the foundations of school mathematics is another. I have selected for review to-day one of the fundamental concepts of mathematics, that of functionality.

May I begin my discussion with a brief historical survey? The idea that the function concept should have a prominent, even a dominant, place in the mathematical programme of the schools may be said to have originated with Klein. Others before him, in France, England and America, as well as in Germany, had advocated the inclusion of " variables " and " functions " in the school programme, but Klein was the first to press the view that " functional thinking " should be made the binding or unifying principle of school mathematics. Klein was equally distinguished as a mathematician and as a leader in the pedagogy of mathematics in Germany, but some years elapsed before his views became generally accepted, even in his own country.

In 1893, in an address before the International Congress of Mathematicians held at Chicago, Klein drew the attention of teachers of mathematics to the vital importance of what he called " functional thinking ". This thesis he developed in a series of conferences for teachers in the years that followed but he seems to have received very little active support. He returned to the attack in 1902, when he published, in a report written in cooperation with Götting, a carefully considered statement of his views. In 1904, before an important conference of mathematicians and scientists held at Breslau, Klein again urged that " the function concept should be the central notion of mathematical teaching and that, as a natural consequence, the elements of the calculus should be included in the curricula of all nine-class schools ". On that occasion Klein impressed the conference so strongly that a special Commission was appointed to formulate definite proposals for reform in the direction indicated by him. The proposals of this Commission were presented to a General Conference of mathematicians held at Meran in 1905, in a form now known as the *Meraner Lehrplan*. These proposals, which are sometimes referred to as " The Charter " of modern mathematics in German schools, set forth the aims of mathematical education in an extremely broad and comprehensive form. That Klein had at last gained his end was evident in the statement that the general aim of school mathematics should be " education in the habit of functional thinking ".

In a Report presented to the International Congress of Mathe-

* A paper read at the Annual Meeting of the Mathematical Association, 5th January, 1934.

maticians held at Rome in 1908, Klein again discussed the concept of functionality and urged that teachers regard it not merely as a mathematical method but as the heart and soul of mathematical teaching. " It is my conviction ", he said, " that the function concept should be the *soul* of mathematical study in the schools ".

For several years before his death in 1925, Klein gave much thought to the practical working out of this theme, with the result that school text-books embodying his ideas began to appear. Among them were some excellent ones by Lietzmann, one of Klein's most distinguished pupils. Klein's great *Elementarmathematik* was a product of the work done at that time.

It is not certain to what degree teachers of mathematics in other countries were influenced by Klein, but it is a fact that at the beginning of the present century a marked change took place in their outlook. Fehr admits that teachers in France were deeply influenced by Klein, but there is evidence that similar doctrines had been preached by French mathematicians, notably Laisant and Darboux, several years before Klein began his campaign. Whether influenced by Klein or not, the work of Laisant, Tannery, Borel and Fehr in France, that of Dintzl, Hŏcevar and Mŏcnik in Austria, that of Veress Pal, Szenes, Mikola and Goldziher in Hungary, that of D. E. Smith, Hedrick, Breslich and Georges in America and that of Perry, Jackson, Godfrey, Siddons and Nunn in England all tended in the same direction. I mention those names merely to show that the movement towards functional thinking has not been confined to one or two countries. It has been a world-wide movement.

The first definite reference in this country to the idea of functionality in school mathematics seems to have been made by Godfrey, in a paper on the Teaching of Algebra read before this Association in 1910. The discussion which followed the paper showed quite clearly that members of the Association were fully alive to the importance of functional thinking and that several of them had given it a prominent place in their own teaching. But it was the publication of Nunn's monumental work on *The Teaching of Algebra* a year or two later that gave the function concept its real place in English education. Nunn's *Algebra* is really a treatise on functional thinking. Reference should also be made to the work of Hedrick in America, which has had such a strong influence on opinion in that country. Hedrick's article on the " Function Concept " in the Report entitled *The National Committee on Mathematical Requirements* (1923) is one of the most eloquent appeals for reform in the teaching of mathematics that has appeared in any language.

We may now ask ourselves : What is meant by the function concept ? In the early history of the subject, even in the early utterances of Klein, the term seems to have been taken to be synonymous with " graphical representation " or " the graphical representation of a function ". Later Klein conceived it much more broadly as the concept of correspondence or dependence, or, more strictly, of " determinate correspondence within a certain domain ". Many writers on this subject take great care to explain that they

do not advocate the teaching of "mathematical functions" or of "function theory"; yet their whole exposition is a treatment of function theory in its most elementary form. It is right that this should be so, for we are slowly coming to realise that most of the fundamental concepts of higher mathematics have their place in the school programme, if only in a rudimentary form. I would go even further and assert that the great concepts of higher mathematics should constitute the framework of mathematical education in the schools. Of these concepts there are four which seem to me to be fundamental to elementary mathematics. They are: (1) the concept of the class; (2) the concept of order; (3) the concept of the variable; and (4) the concept of correspondence—all of which are necessary to an understanding of "a function" or of a "functional relation". I shall not attempt to define these terms. I doubt whether I could define them in such a way as to satisfy the critical audience present. But we may state that two variables, x and y, are in functional relationship when there is a determinate correspondence between the quantities x_1, x_2, etc., of the x variable and the quantities y_1, y_2, etc., of the y variable, the order of arrangement of the quantities being alike.

It may be noted in passing that these four concepts—class, order, variable and function—are also fundamental to modern logic, at least to the logic of life. We cannot reason about anything until we have classified and ordered our material. Let me take a single example of the importance of the concepts "class" and "order" from mental tests. The two propositions "A is taller than B and B is taller than C; therefore A is taller than C" and "B is taller than C and A is taller than B; therefore A is taller than C" differ only in the order of their arrangement; yet one is used as a test of intelligence for seven-year-old children and the other as a test for eight-year-olds.

Let us note also that these same concepts are fundamental to scientific experiment. The scientific worker usually proceeds as follows: He reduces his variables to as few classes as possible, generally two. He then obtains measures of the two variables at a number of convenient intervals, and places them in order. Finally he seeks to determine the law of correspondence subsisting between the ordered quantities.

In recent years several psychologists of note have pointed out the parallel between the procedure of scientific experiment and the act of thinking. Rignano maintains, following the suggestion of Mach, that reasoning is nothing else than a series of experiments performed in the mind and that the logical process of thought is identical with perceptual reality itself. Dewey has made the "thought experiment", conducted in accordance with the scientific method, the basis of his philosophy of "Instrumentalism". His "Complete Act of Thought" is, in essence, an ideal experiment. I suggest that, in the psychology of thinking, mathematical analogies are even more useful than scientific ones. A concept is, in the true mathematical sense, the limit of a class of percepts; the cause and effect relation

may be similarly interpreted. Spearman's "ideal relations" may be expressed in terms of the four concepts of functionality, and even his Noegenetic Principles may be cast into mathematical form.

I shall now proceed with a more detailed account of the application of the notion of functionality to school work, confining myself almost entirely to the elementary field. I shall deal with only the first two years of the secondary school, with pupils who begin the study of secondary school mathematics at the age of about 11 plus. The course that I am about to describe is one that I developed some years ago in an effort to devise a course of "general mathematics".

We begin with simple exercises in tabulation and classification, our purpose being to make the pupil familiar with the attributes or qualities of things and the representation of those attributes or qualities, including that of number, in the form of a classification. So we have denominate classes, such as men and miles, and abstract classes, such as numbers and points. These classes are partitioned, according to our needs, into sub-classes, giving us "manifold" classifications.

The following problem will illustrate the procedure: 160 boys presented themselves for a scholarship examination, the results being published in Grades *A*, *B*, and *C*—*A* being the highest grade. It was found that 26 boys obtained *A* grade and 89 *B* grade in English, and that 31 obtained *A* grade and 85 *B* grade in arithmetic. It was also found that 12 boys had *A* grade and 65 boys had *B* grade in both subjects, while 10 boys had *A* grade in English and *B* grade in arithmetic, and 11 boys had *A* grade in arithmetic and *B* grade in English. The following chart has to be filled in:

English

		A	*B*	*C*	Totals
	A				
Arithmetic	*B*				
	C				
	Totals				160

Then the following questions are asked: (1) How many boys obtained *B* in English and *C* in arithmetic? (2) How many boys obtained *A* in English and did not obtain *A* in arithmetic? (3) How many boys who did not obtain *A* in English did not obtain *C* in arithmetic?

Exercises of this kind are extremely interesting to small boys and give them food for thought.

This discussion of classification leads to the concept of variable classes and to the introduction of a generalised symbol to represent all the quantities of the variable class. Now one of the most im-

portant facts about variables, as I have already indicated, is the order of the quantities of which the variable is constituted. More than one order is, of course, possible to the same variable. Our next business, then, is to present the pupil with sequences (heights, weights, distances, etc.) in which the quantities can be tabulated in some kind of order. Our concern will be with order in value, order in space and order in time, which take us to the beginnings of algebra, geometry and kinematics.

Some teachers object to the introduction of kinematics into elementary mathematics, but to do so is to ignore one of the most fundamental of human intuitions and to set aside a wealth of interesting problem material. As Nunn has said in his *Teaching of Algebra*: " Motion is simply ' geometry plus time ' and any reason which justifies the study of geometry as a branch of mathematics must justify the inclusion of kinematics ". Many of you are familiar with a remarkable paper by William Rowan Hamilton written about a hundred years ago entitled " The Theory of Conjugate Functions with a Preliminary Essay on Algebra as the Science of Pure Time ". In that paper Hamilton says: " The notion or intuition of order in time is not less but even more deep-seated in the human mind than the notion or intuition of order in space. A mathematical science may be founded on the former as pure and as demonstrative as the science founded on the latter ". In this course I have included not only the kinematics of translation but also the kinematics of rotation. There is no reason at all why, when boys are being introduced to the subject of angles, they should not be given problems in angular velocity. They feel that they are in touch with real life when they use the language of the engineer, and speak of the number of " revolutions per second ".

Before I leave the subject of order, may I illustrate one method of introducing " directed numbers " through a number sequence? The boys are taken to the school ground and are asked to " size up, tallest on the right, shortest on the left ", facing the teacher. This arrangement is called an *array*. Some useful statistical terms are then introduced. The middle boy of the array is, of course, the " median ", and the middle boys of the two halves the " quartiles ". It does not matter very much whether boys remember these names, except, perhaps, the median, but they generally do remember them without any prompting. We follow this drill with an exercise in averages, by working out the average height of the array and showing that it does not differ very much from the height of the middle or median boy. Our procedure is to work from the " guessed mean ". The boys are asked to guess their own average and to calculate the deviation of each measure from the guessed average. A very simple computation gives them the true mean. This array also provides us with material for our first exercise in graphical representation, the array of boys naturally suggesting the form that the graph should take. The advantage of beginning graphical work with an exercise of this kind is that the pupil may confine his attention to the changes in one of the variables (the ordinate), since the other variable is a

set of equally spaced intervals. We may note that the number taken by each boy on the command " Number " is a label, which not only marks the boy's position in the array but gives him an identity in his team. If, for example, the master wishes to call out a work party of four, he may call them either by name or by number. Thus numbers become labels of identification. If, instead of numbering the array from the end, as we do to begin with, we number it from the middle, we find that we are in a difficulty, for there are two 2's, two 3's, and so on. The discussion of this difficulty leads us to the notion of positive and negative numbers as identification labels: " Positive I " and " Positive II ", " Negative I " and " Negative II ". Thus the plus and minus signs become component parts of labels used for purposes of identification. At a later stage they become signs of direction, carrying with them the implication of " opposites ", in this case " right " and " left ". In this way three meanings of the plus and minus signs may be very easily illustrated. A little later, not immediately, the use of plus and minus signs as " operators " is developed.

We now proceed to the correspondence between the elements of two classes, of which one-to-one or one-one correspondence is a special case. This leads us to the study of simple analytical functions and correlation, on the one hand, and to geometrical similarity and symmetry, projection and graphical representation, on the other. If one of our classes is a value-class or space-class and the other a time-class, we have, as our functions, *rates* such as velocity and acceleration. If one of our classes is a value-class and the other a space-class, we have statistical or graphical *arrays*. Our most elementary functions will be the formula and the simple equation. Let us note that the formula may be looked upon as the expression not only of a particular relation but also of a general or functional relation. The latter conception is important, but is very often overlooked. The pupil should learn that, while the formula enables him to compute particular values of the " subject " when particular values of the other terms are given, it also enables him to estimate the functional significance of each term entering into the formula. He should learn, for example, that each term has a certain potency or influence on the whole, that a linear term has a potency different from that of a square or cube term. This generalised treatment of the formula should precede any discussion of the equation. As far as possible, I use as materials for this part of the work topics that are likely to have a meaning for the child. I do not bring in complex problems from engineering as is the case in so many text-books but take my material from mensuration, elementary physics, chemistry, economics within the comprehension of the pupil.

The distinction between the particular and the general that I have referred to in relation to the formula must also be emphasised in graphical representation. The graph, like the formula, is a functional whole, representing a general relationship as well as a number of particular relationships. Thirty years ago Perry made a vigorous appeal for more graphical work in schools, but I am quite sure that

he did not intend to subject boys to a laborious plotting of points to obtain results much more easily obtainable by other means. There is very little educational value in such exercises. We do well to remember that the graph is an entity functionally related to the axes of reference and it should be treated as such. The pupil should learn, for example, that the graphical trace of the function $y=x^2$, related to certain axes and to certain units, may do duty for any quadratic equation $y=ax^2+bx+c$. He should learn that the graph of the equation $y-b=2(x-a)^2$ may be readily obtained by two translations of the trace $y=x^2$, one in the positive direction of the x-axis, and the other in the positive direction of the y-axis, and by an appropriate change in the magnitude of the y-unit. Later, he should learn that the wave equation $y=a\sin(x-vt)$ is specifiable as a uniform translation of the wave form $y=a\sin x$ along the x-axis.

Before I leave the subject of graphs, may I urge that "graphs" be not treated as a separate subject or chapter of the mathematical course? I have already indicated one opportunity for the drawing of a graph in the exercise on the variation array. Others occur in connection with the representation of quantities by lines or columns, and with problems in mechanics, when "the line of best fit" is sought. Such exercises are to be found in most English text-books but it is surprising how often they are regarded as skills to be acquired and how seldom as aids to mathematical thinking. It may be noted that statistical arrays, such as the frequency distribution, fall into place in this part of the subject. May I, at this point, urge the inclusion of statistical concepts and methods in the school course? It is surprising how frequently statistical ideas may be introduced into the course of elementary mathematics. We have an opportunity of bringing in the frequency distribution in the results of examinations. As another exercise we may go into the ground and measure the length of the leaves of a tree and make a distribution of them. As a general rule, exercises in the interpretation of graphs should precede exercises in their construction. Teachers would save themselves much labour and annoyance and their pupils much misery, if they would remember that a graph is a story in pictorial form and that children cannot tell their own stories until they have caught the fascination of the stories of others.

I shall now pass on to the teaching of geometry. I suggest that we begin the study of geometry with non-metrical rather than with metrical notions, and that we introduce non-metrical concepts whenever possible in the school course. My first exercise in geometry is a computation of the number of joins of points and the number of meets of lines. In our discussion of formulae in the second or third week of the secondary school course, we verify the formulae $P=\frac{1}{2}L(L-1)$ and $L=\frac{1}{2}P(P-1)$ for the number of intersections of a given number of lines and the number of joins of a given number of points. This work is followed by a study of correspondence, which leads us to similarity, symmetry and conical projection. Thus the general notion of similarity precedes the formal treatment of congruence. Metrical ideas are first introduced in the discussion of

similarity. Field work, with the aid of the plane table, alidade and clinometer, gives us abundant material for the study of similarity and symmetry. In my course of elementary geometry I have followed the suggestions of the Mathematical Association in its Report on the Teaching of Geometry (1923), and have accepted as my fundamental axioms the principle of congruence and the principle of similarity. It is to me a matter of regret that the suggestions contained in that Report have not been more widely accepted. I am convinced that there is no method of treating congruence that is more satisfying to the young mind than that which takes it as a case of one-to-one correspondence. May I add that I take every opportunity of introducing problems in three dimensions even in the first year of school mathematics? Conical and orthogonal projections provide us with some excellent material in this respect.

I must now complete this survey with a few remarks on the teaching of kinematics. Speed or velocity during an interval is defined as the change in position or distance divided by the corresponding change in time. Here note, again, the emphasis on the word "corresponding". There is a correspondence between distance and time intervals which may be clearly expressed in the notation of finite differences. Let us suppose, for example, that the distances traversed by a car in certain time intervals are as follows:

s	t
start	noon
3 ml.	12.5
7 ml.	12.10
12 ml.	12.15

The differences Δs and Δt and the speeds $\dfrac{\Delta s}{\Delta t}$ are:

Δs	s	t	Δt	$\Delta s/\Delta t$
	start	noon		
3 ml.			5 min.	$\dfrac{3\text{ ml.}}{5\text{ min.}}$
	3 ml.	12.5		
4 ml.			5 min.	$\dfrac{4\text{ ml.}}{5\text{ min.}}$
	7 ml.	12.10		
5 ml.			5 min.	$\dfrac{5\text{ ml.}}{5\text{ min.}}$
	12 ml.	12.15		

A discussion of these figures brings out very clearly the change of speed. I am of opinion that the notion of acceleration should be

brought out in this way long before formal exercises are given on the kinematical equations.

In my work I use what I should describe as the Stroud-Picken method of expressing physical units. Stroud, as you are aware, insists that a velocity, for example, be written as $\frac{3\text{ ml.}}{5\text{ min.}}$ rather than as $\frac{3}{5}$ ml. per min. Picken maintains that, since $\frac{1\text{ mile}}{1\text{ foot}} = \frac{5280}{1}$, we should write 1 mile = 5280 . 1 foot = 5280 . foot. I favour this method, with a slight modification, the omission of the dot as the sign of multiplication. In this notation "ft." means "foot" and not "feet". It is the unit of length. Similarly, 1 min. = 60 second and 1 ton = 2240 . pound = 2240 lb.

Note that the notation of finite differences is useful whenever we have sets of corresponding measures. It may be employed, for example, in the analysis of graphs. The similarity of form, of two parabolas, $y=x^2$ and $y=(1-x)^2$, is brought out very clearly when we write down finite differences for corresponding values of the two variables. For example :

$y=x^2$

Δx	x	y	Δy	$\frac{\Delta y}{\Delta x}$
	0	0		
1			1	1/1
	1	1		
1			3	3/1
	2	4		
1			5	5/1
	3	9		
1			7	7/1
	4	16		

$y=(1-x)^2$

Δx	x	y	Δy	$\frac{\Delta y}{\Delta x}$
	0	1		
1			-1	$-1/1$
	1	0		
1			1	1/1
	2	1		
1			3	3/1
	3	4		
1			5	5/1
	4	9		
1			7	7/1
	5	16		

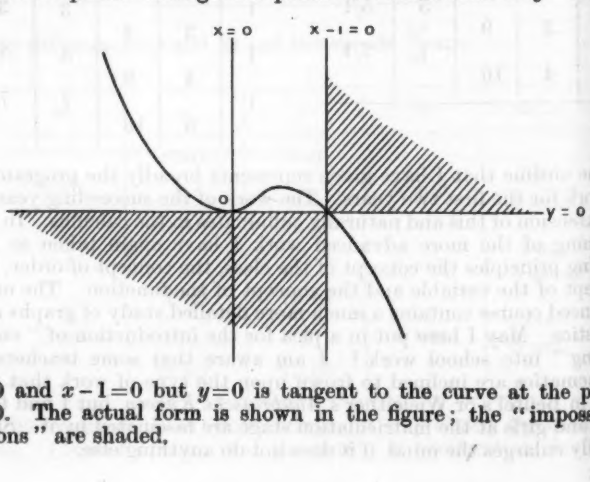
The outline that I have given represents broadly the programme of work for the first two years. The work of the succeeding years is an extension of this and naturally culminates in the calculus. In the planning of the more advanced work I have again taken as my guiding principles the concept of the class, the concept of order, the concept of the variable and the concept of the function. The more advanced course contains a much more detailed study of graphs and statistics. May I here put in a plea for the introduction of "curve tracing" into school work ? I am aware that some teachers of mathematics are inclined to frown upon the type of work that one finds in Beutel's or Wieleitner's *Algebraische Kurven*, but I find that boys and girls at the matriculation stage are fascinated by it. Such a study enlarges the mind, if it does not do anything else.

Mr. Hope-Jones asked if Professor Hamley would give a little more information about the curve tracing to which he had referred.

Professor Hamley: Let us begin by taking a line $y=mx$. This line passes through the origin and, if m is positive, it lies in the first and third quadrants. There are certain regions through which it could not pass, the second and fourth quadrants, for example, because, if the coordinates of any point chosen from the second and fourth quadrants were substituted in the equation there would be a positive sign on one side of the equation and a negative sign on the other. The second and fourth quadrants may be called "impossible" regions. Our field is thus restricted to the first and third quadrants, the axes acting as "sign lines". Arguments of this kind enable us to map out our graph space into "possible" and "impossible" regions.

We now take our old friend $S+\lambda S'=0$, which gives us a line through the points of intersection of $S=0$ and $S'=0$. The graph of the equation $(x-1)(x+2)=(y-2)(y+3)$, for example, passes through the points of intersection of $x-1=0$ with $y-2=0$ and $y+3=0$ and of $x+2=0$ with $y-2=0$ and $y+3=0$. These four lines, which are also "sign lines", give us four points through which the graph passes (describing on diagram). We now have a fair idea of the form of the graph, for we know at least four points through which it must pass and certain regions through which it cannot pass.

Some sign lines are also "tangent lines". For example, $y=x^2$ or $y=x.x$ is a curve through the point of intersection of $y=0$ and $x=0$ (twice). But the line $y=0$ is tangent to the graph at the point $x=0$, $y=0$. Thus when we have a square factor on one side and a linear factor on the other we look for a tangent line. Let us take $y=x^2-x^3=x^2(1-x)$. The "sign lines" are $y=0$, $x=0$, $x-1=0$. The curve passes through the points of intersection of $y=0$ with



$x=0$ and $x-1=0$ but $y=0$ is tangent to the curve at the point $x=0$. The actual form is shown in the figure; the "impossible regions" are shaded.

A further refinement comes in the discussion of "small quantities" to get the shape of the curve at any given point. Thus, if x and y are both "small", we may neglect x^3 in comparison with x^2 and the form of the graph at the origin is "parabolic" ($y=x^2$). Later in the course we learn to transform the axes to any given point as origin and thus to get the form of the graph at that point by a recourse to "small quantities".

Professor E. H. Neville said he would have liked Professor Hamley to give a little emphasis to a point which he sometimes found useful, namely the slight extent to which one's notion of function and variable should be confined to the numerical value. The height of a boy was a function of the boy himself just as much as it was a function of coordinates specifying the position of the boy in the playground. In the case of a triangle, one point was a function of the isogonal conjugate. If one had a point which was varying with the time, describing a curve, the point was itself a function of the independent variable t, and there was no reason at all why one should not write the velocity of the point in the form dO/dt and use that as one's notation for the velocity. When the point O was a function of two variables and was therefore describing a surface, in general one would have the vector equation

$$\frac{\partial^2 O}{\partial u\,\partial v}=\frac{\partial^2 O}{\partial v\,\partial u},$$

which in fact had only to be interpreted to be recognised as two of the fundamental theorems of differential geometry. One did not recognise at once theorems of geometry unless one had the habit, so to speak, of putting the point O there, rather than individual coordinates referred to some axes or other. It would be found that extending the conception of the variable to almost anything one was talking about was a very valuable process of thought.

The President said that Professor Hamley's opening remarks were a remarkable tribute to the amount that could be done by a really eminent mathematician such as Klein in the way of reforming the school curriculum.

Referring to the Δs and Δt in the example given by Professor Hamley when dealing with the subject of kinematics, he supposed that teachers would regard it as essential that the differences in the independent variable should not be equal, as they were in the particular example given.

Professor Hamley agreed.

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977. I am always troubled when I think of my very limited mathematical capacities. It seems as if every well-organized mind should be able to handle numbers and quantities through their symbols to an indefinite extent; and yet, I am puzzled by what seems to a clever boy with a turn for calculation as plain as counting his fingers. I don't think any man feels well grounded in knowledge unless he has a good basis of mathematical certainties, and knows how to deal with them and apply them to every branch of knowledge where they can come in to advantage.—O. W. Holmes, *The Poet at the Breakfast-Table.*

A GRAPHICAL TREATMENT OF ALGEBRAIC EQUATIONS.

By H. Peat.

1. Recent notes in the *Gazette* on the discrimination of the roots of the cubic and quartic suggest that the following method, if not new, is not so widely known as it deserves to be. The only tool required in addition to a facility in reading graphs is a knowledge of differentiation and so the method lies within the scope of the work done by evening technical students and others whose grasp of ordinary algebra is so weak as to prevent their following up the normal methods of discussing the roots of equations. The treatment is also sufficiently general to be of real interest to a type of student whose body of mathematical knowledge is always, from the pressure of other subjects, threatening to disintegrate into a collection of scraps of special methods each applicable to one type of problem. The method yields very easily the discriminants for the quadratic and cubic and it also offers a nomographic device for the solution of numerical equations. In practice, however, even with large scale drawings, students are not able to obtain results as accurate as they can readily get by other graphical methods.

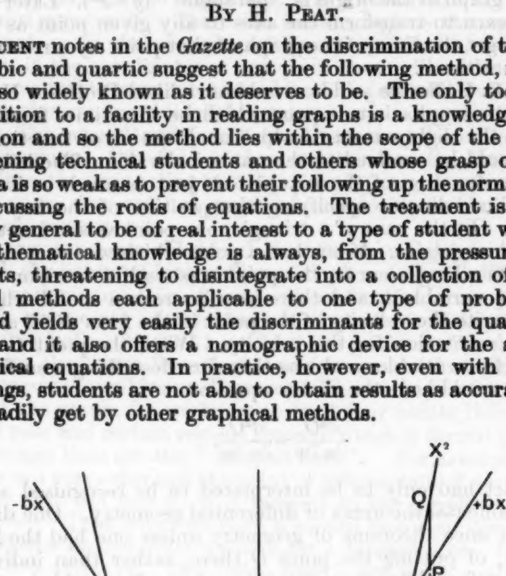

Fig. 1.

2. *The Quadratic.*

The equation $x^2 - bx = -c$ can be solved by drawing $y = x^2$ and $y = bx$ (Fig. 1). If b is positive and c is negative then QP and $Q'P'$ are found, each equal to $-c$, and then Op and Op' are the roots. It is obvious that for every value of $-c$ and of b there are two real roots which have different signs and are outside the range $x = 0$ to $x = b$.

When c is a positive number we take $bx - x^2 = c$ and obtain the roots Or and Ot which are of the same sign and within the range $(0, b)$. In this case there can be two real roots as long as c is less than MN, the maximum intercept between the straight line and the parabola. Hence Om is the root of $d(bx - x^2)/dx = 0$ and $MN = \frac{1}{4}b^2$.

The number of real roots is therefore 2, 1 or 0 according as c is less than, equal to or greater than $\frac{1}{4}b^2$; and so for all cases we have the usual discriminant $b^2 - 4c$.

When b is negative, we may obtain the same result by drawing $y = bx = -b'x$, and consider the intercepts given by $x^2 - (-b'x)$, but it is more exciting for wide-awake students to notice that the rotation of $y = bx$ into the position of negative slope leaves the discriminant unaltered since it involves only b^2.

With a little ingenuity we can illustrate on this diagram all the important things in the theory of quadratics. Thus if the root $Or = \frac{1}{2}b + \alpha$, and $Ot = \frac{1}{2}b - \beta$, we have

$$TG = b(\tfrac{1}{2}b - \beta) - (\tfrac{1}{2}b - \beta)^2 = c,$$

and

$$\beta = \tfrac{1}{2}\surd(b^2 - 4c) = \alpha.$$

Thus the roots of $x^2 - bx + c = 0$ are

$$\tfrac{1}{2}\{b \pm \surd(b^2 - 4c)\},$$

and

$$Ot + Or = b.$$

Moreover if we draw a circle on tr as diameter, then by elementary geometry

$$\begin{aligned} Ot \,.\, Or &= \text{square on tangent from } O \\ &= Om^2 - mt^2 \\ &= \tfrac{1}{4}b^2 - \tfrac{1}{4}(b^2 - 4c) = c. \end{aligned}$$

3. *The Cubic.*

It is sufficient to consider the cubic in the form $x^3 - 3Hx + G = 0$, and we therefore consider $y = x^3$ and $y = 3Hx$ for intercepts of length $-G$. (See Fig. 2, p. 182.)

If G be large and negative while H is positive there can be one real positive root Op, which must exceed $+\surd(3H)$; if H be negative there is only one real root which must be positive and may be zero, as is clear from noting that the straight line $y = -3H'x$ is below the cubic so long as x is positive and above it when x is negative. Similarly if G be positive there must be one negative root whatever the sign of H, the root being less than $-\surd(3H)$ if H be positive also. A cubic has therefore at least one root of sign opposite to the sign of its absolute term.

If G be numerically small there may be two other roots of different sign from the one mentioned above. Thus in the diagram Op, Op' and Op'' are the roots for a small negative G and positive H. The critical value of the intercept which determines whether there shall be one, two or three roots is given by $d(x^3 - 3Hx)/dx = 0$, that is, by $x = \pm\surd H$ and the corresponding values of MN are $\mp 2\surd(H^3)$. The cubic has three real roots if $G^2 < 4H^3$, a pair of coincident roots if $G^2 = 4H^3$, and one real root only if $G^2 > 4H^3$.

The information summarised by the discriminant may be written out at length by reference to the graph and some of the results are shown in the table below.

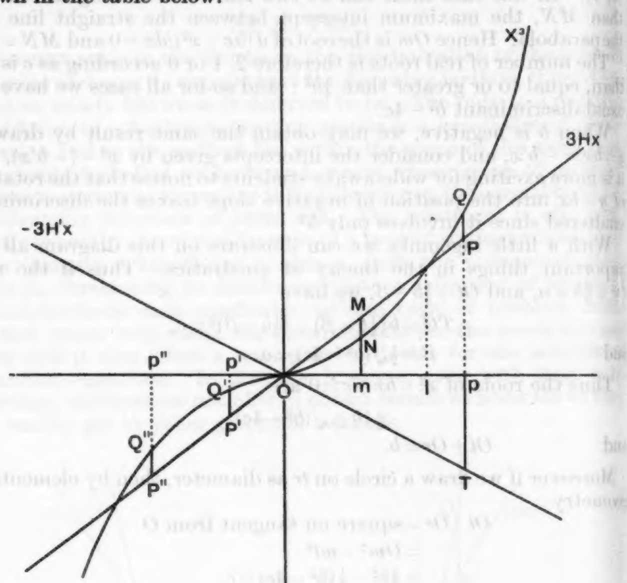

FIG. 2.

Sign of G.	Sign of H.	Numerical relation.	Nature of root and limits between which it lies.
+	+	$G^2 < 4H^3$	1 negative and less than $-\sqrt{(3H)}$ 1 positive and in $0, +\sqrt{H}$ 1 positive and in $+\sqrt{H}, +\sqrt{(3H)}$
+	+	$G^2 = 4H^3$	1 negative and less than $-\sqrt{(3H)}$ 2 positive coinciding with $+\sqrt{H}$
+	+	$G^2 > 4H^3$	1 negative and less than $-\sqrt{(3H)}$
+	−	$G^2 > 4H^3$	1 negative
zero	+	$G^2 < 4H^3$	1 zero : others equal to $\pm\sqrt{(3H)}$
zero	zero		All zero roots

For negative values of G similar results can be written down.

4. *The Biquadratic.*

Before examining this case we notice that if an equation $F(x)=0$ can be written $f(x)-\phi(x)=$ constant, the roots of $F(x)=0$ are given by the appropriate intercepts between the graphs of $f(x)$ and $\phi(x)$. The intercept between the graphs will have maxima and minima where $f'(x)=\phi'(x)$, that is, where the tangents to the curves are parallel. It will therefore be necessary to solve fully an equation of degree $(n-1)$ in order to discuss fully the relations between the roots of an equation of the nth degree. The biquadratic $x^4+6Hx^2-4Gx=p$ always leads to $x^3-3Hx-G=0$, and by using the result for the cubic we have the results that if H be positive there will be one turning point, and if H be negative there will certainly be three.*

The use of Cardan's solution and direct substitution to find the values of MN leads to algebra, apparently more troublesome than is involved in finding the equation of squared differences of the roots of the biquadratic directly, except for the case $H=0$. In this case it is easy to verify that there are generally two real roots except when p is negative and numerically they are greater than $3G^{4/3}$.

The biquadratic may be written

$$(x^4 \pm 6Hx^2) - 4Gx = p.$$

If the positive sign be taken, the diagram is similar to that of the quadratic case (Fig. 1), and there are then always two real roots, except when p is negative and exceeds MN, the point N being found by drawing the tangent which is parallel to the straight line. When H is negative, the diagram is as shown in Fig. 3.

If p be positive there must always be two real roots and there may be four if p is less than the smallest maximum intercept M_2N_2, one root only being positive so long as G also is positive. If p be negative and smaller than M_3N_3 there are four roots (two positive and two negative) while if p is negative and greater than M_1N_1 the biquadratic has no real roots. If $M_1N_1 > p > M_3N_3$ (ignoring the sign), then there must be two positive real roots only. In special cases, of course, roots may coincide at the maximum intercepts.

An alternative but still fundamentally graphical way of finding the three maximum intercepts depends on the following considerations. Since $x^3+3Hx-G=0$ contains the required roots, the graph $G=x^3+3Hx$ may be drawn for positive values of x and then every point on this graph will have for its G-coordinate the particular slope which, with $y=x^4+6Hx^2$ gives a maximum intercept for the corresponding value of x. We can therefore use this graph to find the values of x which yield $G=1, 2, 3, \ldots$. The use of this graph is a more accurate process than the drawing of the tangents parallel to

* It does not appear possible to obtain the formal results discriminating between the roots of the biquadratic by such simple methods as this note uses. Even Professor Dalton's illuminating paper (XVII, No. 224) appears to be defective, for his equation (8) is found by writing down the discriminant for the biquadratic as a preliminary to discussing the case of the biquadratic.

the straight lines. If x_1 be found for a particular G, the intercept is

$$y = x_1^4 + 6Hx_1^2 - 4x_1G$$
$$= -3x_1^2(x_1^2 + 2H).$$

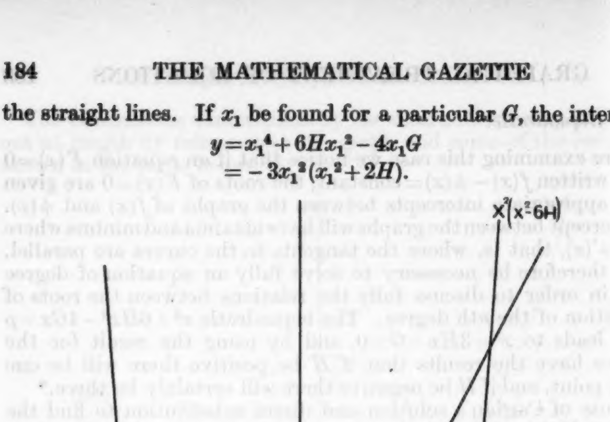

FIG. 3.

The other two critical intercepts are then at the roots of the quadratic

$$(x^3 + 3Hx - G)/(x - x_1) = 0,$$

or

$$x^2 + x_1x + x_1^2 + 3H = 0,$$

and they will exist if

$$x_1^2 > 4x_1^2 + 12H,$$

or if

$$3x_1^2 < -12H,$$

a condition requiring H to be negative. Thus to examine whether a particular biquadratic has one, two, three or four real roots we need only draw $G = x^3 + 3Hx$ and find with any required degree of accuracy the value of x_1. We can then calculate the three intercepts MN, and write down the number of real roots by comparing with p.

5. If with each of the functions $f(x)$ considered above we draw pencils of straight lines through the origin corresponding to integral values of the slopes of the lines, we obtain the nomograms from which rapid but not very accurate solutions may be derived for any member of the family $f(x) \pm kx = c$. H. Peat.

DUNCAN McLAREN YOUNG SOMMERVILLE

It is with deep regret that I record the death of Professor D. M. Y. Sommerville on the 31st January, 1934. Professor Sommerville was a fairly frequent contributor to the *Gazette*, and his work, especially in geometry, was well known. He had been suffering from heart trouble and had been forced to take a rest in September last. However he seemed to make a fairly good recovery and was arranging the work of the 1934 session with me on Tuesday, January 30th. The next day he died at the early age of 54. He leaves a widow for whom much sympathy is felt. He had the satisfaction of finishing his *Analytical Geometry of Three Dimensions* which has just been published by the Cambridge University Press.

As his assistant for eight years I had an opportunity of appreciating his admirable qualities as a teacher, a mathematician, and a man. His early death is an irreparable loss.

F. F. Miles.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

In the May number of the *Gazette* I read with much interest the account of the discussion on "Differentials" which took place at this year's Annual Meeting of the Mathematical Association. As my article on the "Teaching of Differentials" started all the trouble, perhaps I may be allowed to intervene again. I much regret that I was unable to be present at the meeting to join in the discussion, but I am glad to see that although opinions differ I have some supporters in the advocacy of the early teaching of differentials. Personally I welcome the appearance of those elementary text-books whose authors have been bold enough to introduce differentials to schoolboys. It is particularly encouraging to me, as I feel that my advocacy of the early teaching of differentials might be open to the criticism that I am not well enough acquainted with the difficulties of the schoolmaster; but when I see schoolmasters themselves not afraid to mention a "differential" (of the kind in which I am myself interested) I feel that I cannot fairly be charged with a biased opinion on the matter.

In fairness to my namesake I should like to clear up any doubt that there may be as to my identity. One of the speakers made a facetious reference to "two mathematicians having the same name and initials." The blame for that cannot be laid to the charge of either of us, though a friend once told me it was going to cause me a lot of trouble! The two E. G. P.'s have by mutual agreement decided that the Bangorian, who is the writer of this note, shall be known as "E. G. Phillips" and the Oxonian now calls his "mathematical self" Eric Phillips. E. G. P.

University College, Bangor.

ISOTOMICALLY CONJUGATE QUADRILATERALS.

By T. R. Dawson.

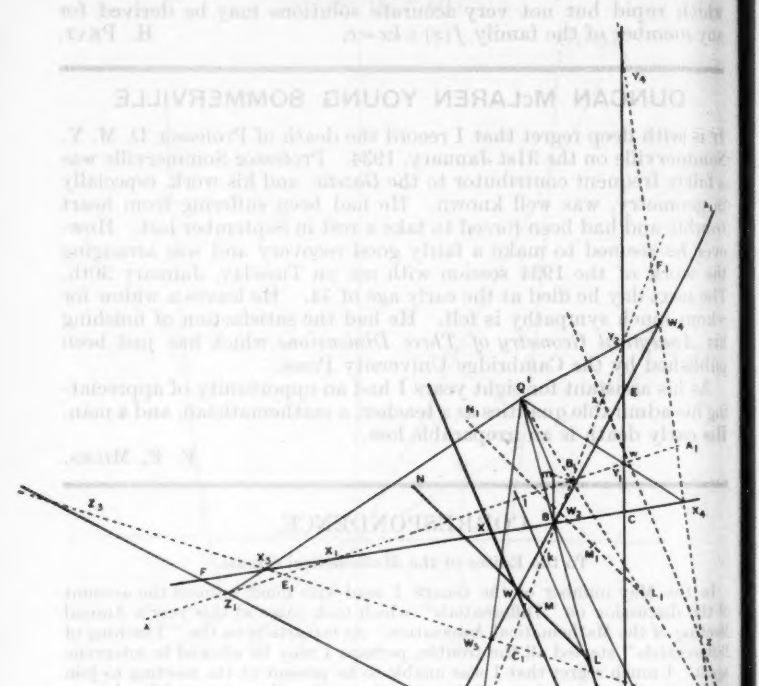

Fig. 1.

If two segments AB, XX_1 of a line have their mid-points coincident, X, X_1 are said to be isotomically conjugate with regard to AB, and

vice versa. For brevity in this paper A, B and X, X_1 will be referred to as "isotoms".

If pairs of isotoms XX_1, YY_1, ZZ_1 are taken on the sides BC, CA, AB of a triangle, and if X, Y, Z are collinear, then by Menelaus' theorem X_1, Y_1, Z_1 are also collinear, and the two lines are said to be isotomically conjugate with regard to the triangle. For brevity pairs of lines related in this way will be termed "isotomic lines". Conversely if two transversals make XX_1, YY_1 isotoms with regard to two sides of a triangle then ZZ_1 will be isotoms on the third side. These simple results, the source of which I have been unable to trace, are the only results on isotomically related figures made use of in the following investigation.

$ABCD$, EF (Fig. 1) is any complete quadrilateral and any transversal meets the four sides in $WXYZ$. Let the isotomically conjugate lines of $WXYZ$ be

$Z_1X_1Y_1$ to triangle CDF;
$Z_2W_2Y_2$ to triangle ADE;
$Z_3X_3W_3$ to triangle ABF;
$Y_4W_4X_4$ to triangle BCE.

Produce these lines to form the new complete quadrilateral $A_1B_1C_1D_1$, E_1F_1. Then along DA, we have $DZ = Z_2A$ by construction,

$$DZ_2 = ZA = FZ_3 \quad \text{and} \quad Z_2A = Z_1F.$$

Thus (A, Z_1), (Z_2, F), (D, Z_3) have a common mid-point and are isotoms. Similar results hold on the other three sides of $ABCD$, EF.

Now let A_1D_1, AD intersect at d. Take $da = DZ$, and join Ba, meeting A_1D_1 at b. Then in the triangle AW_4d, $da = AZ_2$ and $AB = W_4W_2$; hence Bab, $Z_2W_2F_1$ are isotomic lines and $db = W_4F_1$. Again in the triangle FX_4d, $da = FZ_1$ and $X_4B = FX_1$. Thus Bab, $Z_1X_1A_1$ are isotomic lines and $db = X_4A_1$. Hence

$$W_4F_1 = A_1X_4. \quad \text{.............................(i)}$$

Similarly, using the intersection of C_1B_1, CB we obtain $F_1Y_2 = Z_2C_1$. Hence in the triangle X_4FF_1, $X_4A_1 = F_1W_4$, from (i) and $FX_1 = X_4B$. Thus A_1X_1, W_4B are isotomic lines cutting FF_1 at the isotoms λ, λ'. So in the triangle FZ_2F_1, $F\lambda = F_1\lambda'$ and $FZ_1 = Z_2A$. Thus λZ_1B_1, $\lambda' W_4A$ are isotomic lines and $Z_2W_2 = F_1B_1$. But also $Z_2C_1 = F_1Y_2$, above. Therefore (W_2, B_1), (C_1, Y_2), (Z_2, F_1) are isotoms. Similar proofs apply to the other three sides of $A_1B_1C_1D_1$, E_1F_1, whence

THEOREM I. *Each side of either quadrilateral*

$$\text{ABCDEF}, \ \text{A}_1\text{B}_1\text{C}_1\text{D}_1\text{E}_1\text{F}_1$$

is divided by the other three sides of that quadrilateral and the three corresponding sides of the second quadrilateral in three pairs of isotoms.

The eight triads of isotoms are as follows:

On AB: B, W_2; W_3, E; A, W_4. On A_1B_1: B_1, X_1; Y_1, E_1; A_1, Z_1.
On BC: B, X_1; C, X_3; X_4, F. On B_1C_1: B_1, W_2; Y_2, C_1; F_1, Z_2.
On CD: Y_1, E_1; C, Y_2; D, Y_4. On C_1D_1: W_3, E_1; C_1, X_3; D_1, Z_3.
On DA: A, Z_1; Z_2, F; D, Z_3. On D_1A_1: A_1, W_4; X_4, F_1; D_1, Y_4.

It is obvious that we may now select points w, x, y, z on the sides of $A_1B_1C_1D_1E_1F_1$ related to the isotomic triads on those sides exactly as W, X, Y, Z are related to the triads on $ABCDEF$. Moreover, as every three of the points w, x, y, z will lie on a line isotomically conjugate to one of the sides of the original quadrilateral (with regard to a triangle of the new quadrilateral), we see that w, x, y, z lie on one straight line. Hence

THEOREM II. *Taking* w *on* A_1B_1 *so that* $A_1w = B_1Y_1$, *and analogous points* x *on* B_1C_1, y *on* C_1D_1, z *on* D_1A_1, *then* xyzw *is a straight line.*

Next take $Dh = DZ$ on ZD produced, and $Dj = DY$ on YD produced, and let hj (which is parallel to YZ, by similar triangles) cut xB_1 at k, wB_1 at l. Then in the triangle DZ_2Y_2, $Dh = Z_2A$ and $Dj = DY = EY_2$.

Hence hjk, AW_2E are isotomic lines and $Z_2k = Y_2W_2 = C_1B_1$.

Thus $\qquad B_1k = C_1Z_2 = B_1x$ (by construction of x). ....(ii)

And in the triangle DZ_1Y_1, $Dh = Z_1F$ and $Dj = DY = Y_1C$. Thus hjl, EX_1C are isotomic lines, and $Y_1l = X_1Z_1 = A_1B_1$.

Thus $\qquad B_1l = A_1Y_1 = B_1w$ (by construction of w). ....(iii)

From (ii) and (iii) it follows that wx is parallel to kl, that is, to YZ. Hence

THEOREM III. WXYZ, wxyz *are parallel.*

Before proceeding with the main figure we require a lemma. Let ABC (Fig. 2) be any triangle and XYZ any transversal; xyz the isotomic conjugate of XYZ with respect to the triangle ABC.

$$\text{Take}\quad BX_1 = BX,\quad AX_3 = AX,\quad CZ_2 = CZ;$$
$$BY_1 = BY,\quad AZ_3 = AZ,\quad CY_2 = CY;$$

and let X_1Y_1, Y_2Z_2, Z_3X_3 cut xyz in Z_1, X_2, Y_3. Then X_1Y_1, Y_2Z_2 and Z_3X_3 are by construction the reflections of XYZ in B, C, and A respectively, and it will be shown that $X_1Y_1Z_1$, $Z_2X_2Y_2$, $Y_3Z_3X_3$ are all equal to XYZ and similarly divided.

For since $X_1Y_1Z_1$ is isotomic to AzC in the triangle Bxy, $yZ_1 = zx$. Now draw Z_1a parallel to yC and equal to yC and so equal to BY_1. Join aC, aZ. Then aZ_1yC is a parallelogram, and so $Ca = yZ_1 = zx$, and is parallel to zx. Thence $Caxz$ is a parallelogram, and so $ax = Cz = ZA$ and is parallel to ZA. Thence $aZAx$ is a parallelogram, and so $aZ = xA = BX$, and is parallel to BX. Thus in the triangles aZZ_1, BXY_1, $aZ = BX$, $aZ_1 = BY_1$ and the included angles are equal. Hence $ZZ_1 = XY_1$ and is parallel to XY_1, and so the lines $X_1Y_1Z_1$, XYZ are equal, as required.

By isotoms, $yY_2 = yY_1$ and $yX_2 = yZ_1$; thus $X_2Y_2 = Y_1Z_1 = XZ$. And again by isotoms, $xX_3 = xX_1$ and $xY_3 = zX_1$; thus

$$X_3Y_3 = X_1Z_1 = YZ.$$

Hence the *Lemma*: *The reflection in a vertex of a triangle of any transversal is divided by the sides through that vertex and the isotomic conjugate of the transversal into segments equal to those made on the transversal by the sides of the triangle.*

Applying the lemma to the transversal $hjkl$, which is the reflection of YZ in D and also of xw in B_1, we see that

$$hj = YZ, \quad jk = ZW, \quad kl = WX,$$
$$lk = xw, \quad kj = wz, \quad jh = zy.$$

Hence

Theorem IV. *The segments* YZ, ZW, WX *are respectively equal to* yz, zw, wx, *and being parallel, we have further that* wW, xX, yY, zZ *are all equal and parallel.*

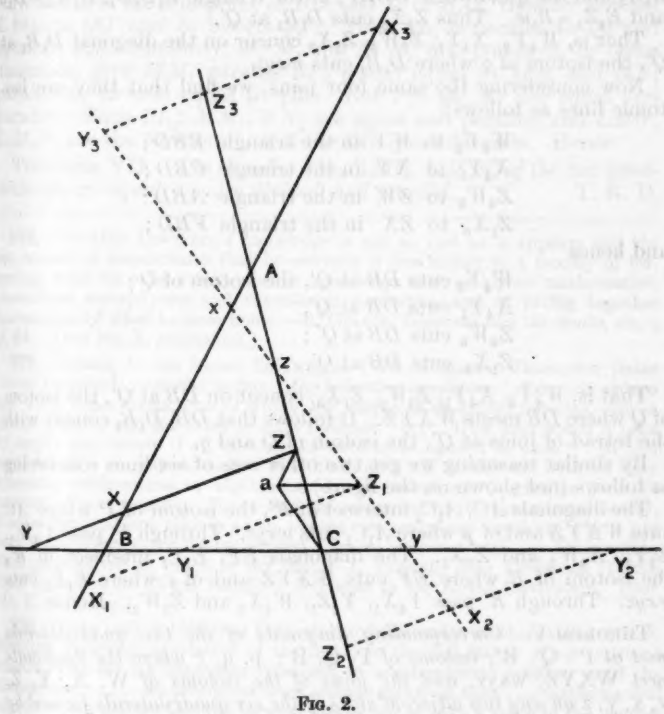

Fig. 2.

It may be noted that there are six lines in all of the type hk, all of which are divided into three segments equal to those on $WXYZ$ and $wxyz$. These are formed by the reflection of $WXYZ$ in the vertices D, A, B, E, F, C of the original quadrilateral and simultaneously by the reflection of $wxyz$ in the vertices $B_1, C_1, D_1, F_1, E_1, A_1$ of the derived quadrilateral.

Next, the six points $ABCDEF$ determine three quadrangles—$ABCD$, convex; $DEBF$, reflex; $AECF$, cross. Consider the iso-

toms of $WXYZ$ on *adjacent* sides of any one of these. Taking W_4Y_2 and the quadrangle $DEBF$, then in the triangle $F_1B_1D_1$, $F_1W_4=D_1z$ and $F_1Y_2=B_1x$. Hence W_4Y_2 cuts D_1B_1 at Q', which is the isotom of q, where xz cuts D_1B_1.

Take now X_4Y_1 and the quadrangle $DCBA$; in the triangle $A_1B_1D_1$, $A_1X_4=D_1z$ and $A_1Y_1=B_1w$. Hence X_4Y_1 cuts D_1B_1 at Q', the isotom of q as in the preceding case. Next take Z_2W_3 and the quadrangle $DCBA$; in the triangle $C_1B_1D_1$, $C_1W_3=D_1y$ and $C_1Z_2=B_1x$; thus Z_2W_3 cuts D_1B_1 at Q' as before. Then taking next Z_1X_3 and the quadrangle $DEBF$, in the triangle $E_1B_1D_1$, $E_1X_3=D_1y$ and $E_1Z_1=B_1w$. Thus Z_1X_3 cuts D_1B_1 at Q'.

That is, W_4Y_2, X_4Y_1, Z_2W_3, Z_1X_3 concur on the diagonal D_1B_1 at Q', the isotom of q where D_1B_1 cuts $wxyz$.

Now considering the same four joins, we find that they are isotomic lines as follows:

W_4Y_2 to WY in the triangle EBD;
X_4Y_1 to XY in the triangle CBD;
Z_2W_3 to ZW in the triangle ABD;
Z_1X_3 to ZX in the triangle FBD;

and hence

W_4Y_2 cuts DB at Q', the isotom of Q;
X_4Y_1 cuts DB at Q',
Z_2W_3 cuts DB at Q';
Z_1X_3 cuts DB at Q'.

That is, W_4Y_2, X_4Y_1, Z_2W_3, Z_1X_3, concur on DB at Q', the isotom of Q where DB meets $WXYZ$. It follows that DB, D_1B_1 concur with the tetrad of joins at Q', the isotom of Q and q.

By similar reasoning we get two other sets of six lines concurring as follows (not shown on the figure):

The diagonals AC, A_1C_1 intersect at P', the isotom of P where AC cuts $WXYZ$ and of p where A_1C_1 cuts $wxyz$. Through P' pass Y_4W_2, Z_2Y_1, X_4W_3 and Z_3X_1. The diagonals EF, E_1F_1 intersect in R', the isotom of R where EF cuts $WXYZ$ and of r where E_1F_1 cuts $wxyz$. Through R' pass Y_4X_1, Y_2Z_1, W_4X_3 and Z_3W_2. Hence

THEOREM V. *Corresponding diagonals of the two quadrilaterals meet at* P′, Q′, R′, *isotoms of* P, Q, R; p. q, r *where the diagonals meet* WXYZ, wxyz, *and the joins of the isotoms of* W, X, Y, Z, w, x, y, z *on any two adjacent sides of the six quadrilaterals formed by* ABCDEF, $A_1B_1C_1D_1E_1F_1$—*twelve joins in all—concur in fours in* P′, Q′, R′.

Now draw Bm parallel and equal to CY_1. Join mQ', mB_1. Then BmY_1C is a parallelogram and so $mY_1=BC=X_3X_1$ and is parallel to X_3X_1. Therefore $Y_1mX_3X_1$ is a parallelogram and so

$$mX_3=Y_1X_1=B_1E_1$$

and is parallel to B_1E_1. Therefore $B_1mX_3E_1$ is a parallelogram, and so $mB_1=X_3E_1=D_1y$ and is parallel to D_1y. Thus in the triangles

BmQ', DYQ, $BQ'=DQ$, $Bm=Dy$ and the included angles are equal; hence $YQ=mQ'$. And in the triangles B_1mQ', D_1yq, $B_1Q'=D_1q$, $B_1m=yD_1$ and the included angles are equal; hence $yq=mQ'$. That is, $yq=mQ'=YQ$. Therefore Q, q divide $WXYZ$, $wxyz$ in two new equal segments, and Qq is parallel to Ww. Similarly for Pp, Rr. Hence

THEOREM VI. *Corresponding diagonals of the two quadrilaterals cut off equal segments* PQRWXYZ *and* pqrwxyz *on the* WZ, wz *lines and give three joins* Pp, Qq, Rr *equal and parallel to* Ww, Xx, Yy, Zz.

Now if L is the mid-point of BD, and L_1 of B_1D_1, then by isotoms L bisects QQ' and L_1 bisects qQ'. Thus LL_1 is parallel to Qq and equal to $\frac{1}{2}Qq$. Similarly, M, M_1 and N, N_1, mid-points of the other diagonals, give MM_1 parallel to Pp and equal to $\frac{1}{2}Pp$, and NN_1 parallel to Rr and equal to $\frac{1}{2}Rr$. But Pp, Qq, Rr are equal and parallel. Thus LL_1, MM_1, NN_1 are equal and parallel, and LMN, $L_1M_1N_1$ are the Newton lines of the two quadrilaterals. Hence

THEOREM VII. *The Newton lines* LMN, $\mathrm{L_1M_1N_1}$ *of the two quadrilaterals are equal, equally divided, and parallel.* T. R. D.

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978. Probably Lawrence's knowledge is not so vast as it appears and the impression of omniscience that he conveys is due rather to a faculty of forgetting what he calls utterly useless knowledge such as higher mathematics, class-room metaphysics and theories of aesthetics, and of fitting together harmoniously what he does know.—R. Graves, *Lawrence and the Arabs*, ch. i. p. 24. [Per Mr. J. Buchanan.]

979. Writing to his friend Baldwin in 1773, Benjamin Thompson (later Count Rumford) proposed to him " the following question: ' A certain cistern has three brass cocks, one of which will empty it in fifteen minutes, one in thirty minutes and the other in sixty minutes. Qu.: How long would it take to empty the cistern if all three cocks were to be opened at once ? ' If you are fond of a correspondence of this kind, and will favour me with an easy question, arithmetical or algebraical, I will endeavour to give as good an account of it as possible. If you find out an answer to the above immediately, I hope you will not take it as an affront my proposing anything which you may think so easy, for I must confess I scarce ever met with any little notion that puzzled me so much in my life."—J. Tyndall, *New Fragments* (1892), from a lecture on Count Rumford delivered in the Royal Institution. [Per Mr. E. G. Hogg.]

980. It was only in 1851 that Eton elevated the senior mathematical master, Stephen Hawtrey, a relative of the head, to the full status of an assistant master, and his six assistants remained in a subordinate position until the Commission. . . . So persistent were archaic survivals that . . . mathematical masters were neither allowed to wear gowns nor to take a share in the general discipline of the school, received lower salaries and could not become house-masters! The attitude of headmasters towards mathematics is illustrated by a story of an interview between a newly-appointed mathematical master and his chief. The assistant's attempts to extract from the head any expression of opinion on the mathematical syllabus were cut short by the brief answer, "That's as you please"; and, when he went on to make enquiries as to his status and disciplinary powers, as for instance whether the boys would be expected to cap him, he received the equally curt reply, "That's as they please".—*Secondary Education in the Nineteenth century*, ch. iii. [Per Mr. P. J. Harris.]

BROADENING THE BASIS OF STUDY IN ARITHMETIC.

By H. M. Cook.

Arithmetic has suffered in the past, and still suffers, from being regarded as merely a tool subject, without interest or content of its own. The three R's have a very definite meaning for many an English business man of to-day, who himself had little or no schooling after the age of 14 : he requires of his would-be employees that they shall be able to read print, to write a legible hand, to add up a column of figures and make certain specialised mental calculations, chiefly concerned with our complicated monetary system.

The University Honours graduate who teaches in our Secondary Schools tacitly admits that these rudiments are necessary, but is in a hurry to turn the reading lesson into literary appreciation, writing into self-expressive composition, and arithmetic into the various branches of mathematics, or the hand-maid of science.

Now the power to read and write as judged by the man in the street is no worse for having been exercised on the subjects favoured by the academic teacher of English, but considerable facility in algebra and geometry is as a matter of fact compatible with inaccuracy, and even helplessness, in dealing with the arithmetic required in financial business.

For this reason it has become traditional in our schools that a large amount of information on business practice shall be given by the teacher of Arithmetic, and the subject as it appears in our school time-tables and text-books is not a branch of mathematics in any true sense, so that the mathematically enthusiastic teacher often regards it as a nuisance—it must be taught because it will be examined on in the School Certificate, but it is drudgery both to teacher and pupil to be got over somehow and in no way to be made interesting.

The class teacher in the elementary school may also consider the daily arithmetic lesson as a piece of routine, not requiring much thought in preparation, and therefore an " easy " subject, but having little interest or cultural value in itself.

Now surely the root of this unsatisfactory attitude where it exists—(and I have no wish to suggest that it is universal)—is that the teachers themselves are not clear-minded about the fundamentals of arithmetic—they do not appreciate that the word really covers two subjects : the pure Arithmetic of the study of number, which is one of the bases of all mathematical work, (another being the study of space, with geometry as its primary form) ; and, the application of Arithmetic to various customs in the civilised, and particularly the financial, world, and to calculations in connection with scientific measurement.

Pure arithmetic is international, applied arithmetic involves measurement, and as long as different nations use different units of measurement, whether of goods or of value, this branch will have to be taught nationally.

My complaint is that the text-books print their chapter headings

as " Decimals ", " Interest " in exactly the same type, and neither teacher nor pupil realises the distinction between pure and applied arithmetic. They do indeed distinguish between sums and problems, but merely as a matter of wording and arrangement of work and often without appreciating the fundamental differences between the two branches—in fact they would include the compound rules and even simple and compound interest among the sums, while classifying stocks and shares as problems, if indeed they do not demand that these last should be banished from the schools altogether. They have, of course, long been a class-room subject and little more : in the past probably not one pupil in ten remembered enough of what he learnt about them at school to be of any value to him by the time he had money to invest, but Government borrowing during the war did serve to make some of our teaching a little more real. It is, however, still a moot point whether information about financial practice should necessarily be given as part of Arithmetic, but with Stock Exchange reports filling so large a proportion of our daily newspaper as they do, and national and international indebtedness affecting the lives of all, it should be included in the general education of every schoolboy and schoolgirl. Where economics and civics are separate subjects, or treated as part of history, the arithmetic teacher may perhaps leave Stocks and Shares and Citizenship Arithmetic generally to his colleague, but if his is the only opportunity to give this knowledge he should not shirk it even to open up vistas of mathematical thought to the brighter pupils.

Not that I would imply that financial and commercial practice is the only or even the chief application of arithmetic to which attention should be paid in schools, but the mathematical enthusiast is less likely to neglect areas and volumes, graphs, the reading of tables, and the approaches to science. Moreover, everyone will agree that modern developments in mechanical transport and communication, in electricity and wireless, have added to the urgency of the problem, but they did not create the necessity, which already existed, of attempting to awake in every child a clear conception of number and to supply him with enough knowledge to apply numerical work intelligently to the daily life around him.

If the teacher is conscious of this double aim the Arithmetic lesson can become vividly human and interesting, but if he loses sight of either aspect the pupil inevitably regards each sum as a separate routine task which he will perform lifelessly and mechanically.

The word " human " has been used advisedly, for a part of the trouble is that the present-day specialist in mathematics has so often had a scientific, as distinct from a humanistic, training, and is apt to stress mechanical dealing with written figures while paying but little attention to the need for real familiarity with numbers.

Number in the abstract is a human subject—it is one of the highest, if not the highest, development of the human power of thought, but this development has been achieved little by little through continual interaction with the concrete, and the child, like the race, must

acquire power over number by using it in daily life—in counting and measuring the objects around him in school and home—in computing costs and making payments—in reading numerical statements—in visualising larger and larger quantities—and smaller and smaller quantities too. No custom which involves the use of number should be despised by the teacher who has himself any conception, however inadequate, of the abstract study of number: but the teacher and the pupil who by nature prefer the abstract are comparatively rare—the majority want to know the practical use to which every type of calculation can be put, and it is in the applications of number that they recognise a human interest.

When we conceive of the teaching of number in this broad way it will matter very little to us that each of the "application channels" through which we work has its own terms, which it uses in a fashion peculiar to itself—the physicist may use "solid" as distinct from "fluid", the geometer considers a "solid" as a defined portion of three-dimensional space, the man in the street contrasts "solid" with "hollow", yet no confusion results. Again, the practical engineer, speaking of the gradient of a road, has in mind a ratio different from that by which the mathematician defines the gradient of a graph. Similarly we can by careful attention to the context master the meaning of sentences containing that elusive phrase "per cent.". If the analyst is entitled to say that 60% of the quantity of liquid before him is water, the merchant may equally well say of the money in his till that 15% is profit: if the rod in the laboratory experiment expands ·01%, the cost of living can go up 3%: if it can be correctly stated that a population has been reduced 4% by emigration, it is equally possible to allow the reduction of a bill by a $2\frac{1}{2}$% discount.

Terms have no meaning until they are defined, and mathematicians have no monopoly of the right to define terms except in the domain of pure mathematics—it is sheer arrogance on their part to give their blessing to compound interest because they have discovered that a study of financial practice in this matter is a help in teaching the mysteries of the exponential theorem, but to condemn Banker's Discount by insisting that they as mathematicians have a truer method of calculating what should be deducted. If teaching on these subjects or on Profit and Loss is to be included in the Arithmetic course at all, the terms used must be those current in commercial circles.

I venture to suggest that teachers might well try to gain a new vision of the immense range and possibilities of Arithmetic—to stand back and try to see the forest whole before they plunge into it and find their view obscured by the trees. H. M. C.

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981. A mathematician who solves a profound problem, a poet who creates an image of beauty in the mind that was not there before, imparts knowledge and power to others, in which his greatness and his fame consists, and on which it reposes. Jedediah Buxton will be forgotten; but Napier's bones will live.— W. Hazlitt, *The Indian Jugglers*.

MATHEMATICAL NOTES.

1104. *A problem in elementary probability.*

(1) A batsman plays a large number of innings for which his average number of scoring strokes is a. There is a certain constant probability that he gets out to any ball bowled to him, and another constant probability that he makes a scoring stroke (and so a third constant probability that neither of these events happens). Prove that the commonest number of scoring strokes in an innings is 0, and, more generally, that the number of innings of exactly n scoring strokes is proportional to

$$\left(\frac{a}{a+1}\right)^n.$$

It may be assumed that all innings are completed and that the batsman is never run out.

(2) Discuss the assertion that "a batsman's average is governed primarily by his very large scores".

(3) Show that if we neglect the differences in the scoring values of different strokes, and psychological or other factors peculiar to the game of cricket, then a batsman who has a large average a should score $2a$ or over in about e^{-2} of his innings.

(4) Discuss the application of the preceding results to the case of J. B. Hobbs.

[Hobbs has, to a first approximation, scored 50,000 runs in 1000 innings, failed to score 50 times, made 200 centuries and 15 double centuries: more precise details will be found in the appropriate works of reference.] G. H. Hardy and J. E. Littlewood.

1105. *Legendre again.*

The bibliographical story of the *Exercices de Calcul Intégral* and the *Traité des Fonctions Elliptiques* in the *Gazette* for July 1933 (vol. 17, p. 200) is not complete, for it does not begin at the beginning. There are two distinct versions of title-page to the *Exercices*, which both bear the date 1811; on one the author's name appears as Le Gendre and the volume is described as "Tome premier", while on the other the name has the more familiar form and there is nothing to suggest that the volume is not complete in itself. The explanation is that the author did not think of the volume of 1811 as an instalment when he published it. "Le volume", he says in the *Avertissement* to the second volume, ". . . fut suivi d'un Supplément à la première partie, qui parut au commencement de 1813. Je regardais alors l'Ouvrage comme terminé, et je ne pensais guère à lui donner une continuation." That even the *Quatrième Partie*, June 1814, was treated, at least by Mme. Ve. Courcier, the publisher, as a further supplement to the volume of 1811, is proved by her list of *Ouvrages Nouveaux, 1812-1814*, happily preserved in a copy of the volume which came to the Royal Astronomical Society from the old Spitalfields Mathematical

Society ; in this list occurs the entry

Legendre ... Exercices ..., in-4, avec deux Supplémens, 28 fr.

Les deux Supplémens, imprimés en 1814, se vendent séparément, 9 fr.

However, there would seem to be a mistake, since there is the evidence both of Legendre as quoted above and of the Catalogue of the Bibliothèque Nationale, that the first Supplement appeared in 1813. Some confirmation that the two Supplements were added to the volumes in stock without any change in the title-page is provided by the copy just mentioned, which has precisely this composition.

The decision to cast the volume of 1811, enlarged only by the Supplement of 1813, for the new *rôle* of "Tome premier" was taken therefore between June 1814 and July 1816, when the first part of "Tome troisième" appeared; it may be assumed that the new title-page, though dated 1811, belongs to this period, and it is unlikely that it was introduced before the Cinquième Partie, which appeared in August 1815, was being written. That is to say, this revised title-page should be assigned to 1815 or 1816, in spite of the date it bears. The spelling of the author's name is not a clue.

E. H. N.

1106. *Stop. Caution. Go.*

We are accustomed to write

$$2\cos n\theta = c^n - nc^{n-2}/1! + n(n-3)c^{n-4}/2! - n(n-4)(n-5)c^{n-6}/3! + \dots, \quad \text{.........}(1)$$

where $c=2\cos\theta$; it is assumed that n is a positive integer, and the series comes automatically to an end. The coefficients in this expansion appear in another series: if $x=t(1-t)$, then, as we find for example in Bromwich, *Infinite Series*, (2 ed.), p. 199, Ex. 16,

$$(1-t)^n = 1 - nx/1! + n(n-3)x^2/2! - n(n-4)(n-5)x^3/3! + \dots, \quad \text{...}(2)$$

where no restriction is placed on n. At first glance this latter result appears definitely to be wrong. For $n=4$ the polynomial on the right becomes $1-4\{t(1-t)\}+2\{t(1-t)\}^2$, in which t^4 has the coefficient 2, and for $n=5$ the polynomial is $1-5\{t(1-t)\}+5\{t(1-t)\}^2$, in which t^5 does not occur.

The mistake is one of interpretation, rendered natural by familiarity with (1). In the series in (2), the factor $n-8$, let us say, does not occur in all the coefficients from that of x^5 onwards, but only in those of x^5, x^6, x^7, and when we put $n=8$ in this series, we do not reduce the series to the polynomial

$$1-8x+20x^2-16x^3+2x^4; \quad \text{.........................}(3)$$

we have still an infinite series, but there is a gap after the polynomial, and the series resumes in the form

$$-x^8(1+8x/1!+8.11x^2/2!+8.12.13x^3/3!+\dots). \quad \text{..........}(4)$$

Now by the expansion (2) itself, the value of this expression is $-x^3(1-t)^{-3}$, that is, $-t^3$, and it follows that the value of the polynomial (3) is $t^3+(1-t)^3$, as can be verified immediately. More generally, if in (2) the exponent n has a positive integral value, a gap of $\frac{1}{2}n-1$ or $\frac{1}{2}(n-1)$ terms appears in the series on the right; this gap separates a polynomial which is equal to $t^n+(1-t)^n$ from an infinite series whose value is $-t^n$, and (2) remains valid as an expansion of $(1-t)^n$.

Perhaps the simplest way of establishing (2) is by means of a differential equation. If $x=t(1-t)$ and $y=t^n$, it is easily seen that y satisfies the equation

$$x(1-4x)y''-\{(n-1)(1-4x)-2x\}y'-n(n-1)y=0, \quad\text{......(5)}$$

and since the substitution of $1-t$ for t leaves x unaltered, it follows that $(1-t)^n$ satisfies the same equation, and that the general solution of the equation is $At^n+B(1-t)^n$. By formal development, if

$$y=a_0+a_1x/1!+a_2x^2/2!+a_3x^3/3!+\dots,$$

the coefficients are related by the sequence of formulae

$$\begin{aligned}(n-1)a_1+n(n-1)a_0 &=0,\\ (n-2)a_2+(n-2)(n-3)a_1 &=0,\\ (n-3)a_3+(n-4)(n-5)a_2 &=0,\\ \dots\dots\dots\dots\dots\dots &\;\dots\dots\dots\text{(6)}\end{aligned}$$

If n is not a positive integer, this sequence determines every coefficient uniquely in terms of a_0; with the same condition, t^n, which resembles x^n near $x=0$, cannot be developable in positive integral powers of x, and therefore the solution given by the series, with $a_0=1$, can be nothing but $(1-t)^n$. If, however, n is a positive integer, the argument fails at two points. The sequence determines the coefficients only as far as the term in $x^{\frac{1}{2}n}$ or $x^{\frac{1}{2}(n-1)}$, and all subsequent coefficients may be zero; that is, the differential equation has a solution which is a polynomial in x. This is the polynomial found by treating the series in (2) as coming to an end. But since now t^n is expansible as a power series in x, all we are entitled to say is that this polynomial is equal to $(1-t)^n+At^n$ for some value of A; direct examination of the highest power of x in the polynomial shows that $A=1$. To recover the expansion (2) in its earlier sense, we observe that while we *can* satisfy the sequence (6) by supposing $a_k=0$ for all values of k beyond $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, the sequence actually *imposes* zero values only as far as $k=n-1$. Beyond this point there are no vanishing terms in the sequence, and we can therefore determine the subsequent coefficients uniquely as multiples of an arbitrary a_n. Thus we find a solution which is independent of the polynomial, and since this solution is zero when $t=0$, it is a multiple of t^n.

It is of some interest to observe that whereas in (2) the gap is accidental and the coefficients on the two sides of the gap are organically related, when we develop the polynomial and the infinite

series from the sequence (6) the gap in the sequence severs them completely from each other.

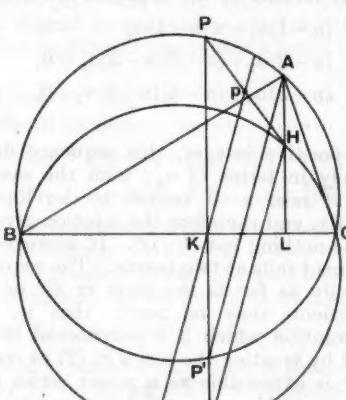
Professor Watson reminds me that to avoid the very ambiguity encountered here he introduced in his *Bessel Functions* (p. 103), two special symbols, a formalised signal indicating whether the series is to go on, ⌉, or to stop, ⌉.

E. H. N.

1107. *The envelope of the Simson line.*

It is interesting to envisage the envelope of the Simson line (*Gazette*, XVII., No. 226, p. 291) from a purely geometrical point of view. I send a note on this at Professor Lodge's request:

H being the orthocentre, and PK (as in his figure) the perpendicular from a point P on the circumcircle to the base of the triangle ABC, meeting that circle in P' and the equal circle BHC in P'', he begins by noting that the Simson line of P is parallel to AP'. Hence (or similarly) it is parallel to HP''. Since it passes through K, the middle point of PP'', it also passes through p, the middle point of PH.



Now the locus of p is the nine-point circle, H being the centre of similitude of it and the circumcircle; and as P, p move round their respective circles, P'' will move (correspondingly) backwards round its circle, and therefore HP'' will rotate backwards at *half* the angular rate. So pK, being parallel to HP'', rotates backwards at half the angular rate at which p rotates about the centre of its circle.

Simple geometrical properties of the hypocycloid at once identify the line in question with a tangent to a curve of this species, touching the nine-point circle. Since the Simson line will rotate back-

wards through 60°, while p changes its direction by 120° (involving a relative change of direction of 180°) the line, and its envelope, will touch the nine-point circle three times during a complete revolution; at the intermediate positions the line will be perpendicular to the circle and will be the tangent at a cusp; *i.e.* we have a 3-cusped hypocycloid.

Since two near positions of the line will be inclined at *half* the angle between the corresponding radii to p, the distance of a cusp from the nine-point circle will be *twice* the radius of that circle. This fits the fact, otherwise obvious, that the hypocycloid will be that generated by the rolling of a circle equal to the nine-point circle inside a circle of 3 times the diameter.

P. J. HEAWOOD.

1108. *The envelope of the Simson lines of a triangle.*

Philip Franklin of the Massachusetts Institute of Technology has a paper on this topic in the *Journal of Mathematics and Physics*, Vol. VI, No. 1 (November 1926), pp. 50-61. In this article he advances over some of the ground covered by Professor Lodge in the *Gazette* for December 1933, and finishes by showing the relation between the Steiner hypocycloid and the Morley triangle, where by the latter he means the equilateral triangle determined by certain of the intersections of the internal trisectors of the angles of a triangle. (The *Gazette* gives it the same name.) It seems fitting that there should be some relation when two constructions start from any triangle and lead to figures having perfect three-way symmetry.

University of Michigan. N. ANNING.

1109. *The triple vector product.*

The following method of obtaining the expansion of a triple vector product is, I think, a trifle shorter than that given by Professor Neville in Note 1089 (December, 1933). We start, as he does, by noting that

$$[[\mathbf{ab}]\mathbf{c}] = x\mathbf{a} + y\mathbf{b},$$

where x and y are two scalar quantities. Multiply both sides of this equation scalarly by a vector $\mathbf{f}$, which lies in the plane of $\mathbf{a}$ and $\mathbf{b}$ and is perpendicular to $\mathbf{b}$. Then

$$x(\mathbf{af}) = ([[\mathbf{ab}]\mathbf{c}]\mathbf{f})$$
$$= (\mathbf{c}[\mathbf{f}[\mathbf{ab}]]).$$

But, if θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $[\mathbf{f}[\mathbf{ab}]]$ is a vector of magnitude $abf \sin\theta$ along $\mathbf{b}$, and hence

$$[\mathbf{f}[\mathbf{ab}]] = (\mathbf{af})\mathbf{b}.$$

Thus $$x(\mathbf{af}) = (\mathbf{cb})(\mathbf{af})$$

and $$x = (\mathbf{cb}).$$

In the same way $$y = (\mathbf{ac}).$$

Incidentally the result can be written in the symmetrical form,

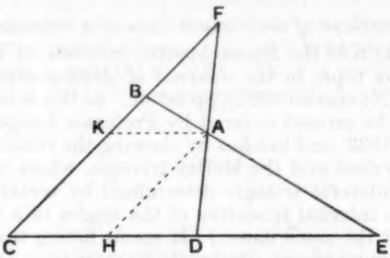
$$[[ab]c] = \begin{vmatrix} (ac), & (bc) \\ a, & b \end{vmatrix}.$$

This proof was suggested to me by another proof given by Mr. O. F. T. Roberts of Aberdeen University.

H. V. LOWRY.

1110. ***The middle points of the three diagonals of a complete quadrilateral.***

$ABCD$ is a quadrilateral; CD and BA are produced to meet at E, CB and DA produced to meet at F. Draw AH parallel to BC to meet CD at H, AK parallel to DC to meet CB at K.



Then
$$\begin{aligned} HD/HE &= (HD/HA)\,.\,(HA/HE) \\ &= (KA/KF)\,.\,(KB/KA) \\ &= KB/KF. \end{aligned}$$

Hence H and K, D and B, E and F may be regarded as simultaneous positions of two equal particles travelling uniformly along HDE and KBF; their mass-centre also travels uniformly along a straight line. Therefore the middle points L, M, N of HK, DB, EF are collinear. But the middle point of HK is also the middle point of CA. Therefore the middle points of CA, DB, EF are collinear.

W. J. DOBBS.

1111. ***The curvature of a plane curve.***

1. The curvature of a curve whose polar equation is given is usually obtained by differentiating the equation

$$\psi = \theta + \text{arc tan}\left\{ r \bigg/ \frac{dr}{d\theta} \right\},$$

or else, by a purely analytical process, from the well-known formula

$$\kappa = y_2/\{1+y_1^2\}^{\frac{3}{2}}.$$

The following method does not appear in the usual text-books. We denote differentiation with respect to θ by suffixes so that

$$\frac{d}{d\theta} \equiv s_1 \frac{d}{ds}.$$

Then differentiating twice the equations

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we have the four equations

$$s_1 \cos\psi = -r\sin\theta + r_1\cos\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$s_1 \sin\psi = r\cos\theta + r_1\sin\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$$s_2\cos\psi - \kappa s_1^2 \sin\psi = -r\cos\theta - 2r_1\sin\theta + r_2\cos\theta, \quad \ldots\ldots\text{(iii)}$$

$$s_2\sin\psi + \kappa s_1^2\cos\psi = -r\sin\theta + 2r_1\cos\theta + r_2\sin\theta. \quad \ldots\ldots\text{(iv)}$$

We now "cross-multiply" these equations, that is, subtract the product of (ii) and (iii) from the product of (i) and (iv), and we have at once

$$\kappa s_1^3 = r^2 - rr_2 + 2r_1^2,$$

whence $\kappa = \{r^2 - rr_2 + 2r_1^2\}/\{r^2 + r_1^2\}^{\frac{3}{2}}$.

The mechanical effectiveness of the method is more apparent if equations (i) and (ii) can be written on a single line, with equations (iii) and (iv) on a single line underneath them.

2. Similarly, when the Cartesian equation of a curve is given in the implicit form $\phi(x, y) = 0$, we may use Monge's notation and write $h = \{p^2 + q^2\}^{\frac{1}{2}}$. Then

$$h\cos\psi = q, \quad h\sin\psi = -p,$$

and differentiating with respect to the arc σ,

$$-h\kappa\sin\psi + h_1\cos\psi = dq/d\sigma, \quad h\kappa\cos\psi + h_1\sin\psi = -dp/d\sigma.$$

Cross-multiplying,

$$\begin{aligned} h^2\kappa &= -(q\,dp - p\,dq)/d\sigma \\ &= -\{q(r\,dx + s\,dy) - p(s\,dx + t\,dy)\}/d\sigma \\ &= -\left\{(qr - ps)\frac{q}{h} - (qs - pt)\frac{p}{h}\right\}, \end{aligned}$$

and $\kappa = -\{rq^2 - 2p\,qs + tp^2\}/(p^2 + q^2)^{\frac{3}{2}}$.

3. It is an instructive example on partial differentiation to obtain the coordinates of the centre of curvature by differentiating the equation to the normal,

$$(X - x)q - (Y - y)p = 0,$$

whence

$$(X - x)\frac{qs - pt}{h} - (Y - y)\frac{qr - ps}{h} - \frac{p^2 + q^2}{h} = 0,$$

and

$$X = x - \frac{p(p^2 + q^2)}{rq^2 - 2p\,qs + tp^2},$$

$$Y = y - \frac{q(p^2 + q^2)}{rq^2 - 2p\,qs + tp^2}.$$

G. H. LESTER.

REVIEWS.

The Nature of Mathematics. By M. Black. Pp. xiv, 219. 10s. 6d. 1933. (Kegan Paul)

The Principles of Logic. By C. A. Mace. Pp. xiii, 388. 12s. 6d. 1933. (Longmans)

Controversies on the foundations of mathematics show no signs of ending in agreement, and three main schools of thought exist at present: the followers of the logistic method (*Principia Mathematica*) regard mathematics as a branch or continuation of logic; the formalists (Hilbert) look on it as the science of the formal structure of symbols; the intuitionists (Brouwer) found mathematics on a basic intuition of the construction of an infinite series of numbers. The task of giving an account of the three points of view is extremely difficult, for, as has often been remarked, adherents of different schools speak different languages and intercommunication is almost impossible; the advocate of one theory, in attempting to understand another, distorts its meaning in the effort to grasp it; and, as none of the theories can be *proved*, a particular one can only be accepted because of its appeal, because it accords with one's prejudices or general philosophy.

Each method has its disadvantages: the logistic method seems too much bound up with the world, as when it is stated that the value of $5+1$ depends on whether there are five individuals in the world or more than five; the doctrine of types makes it difficult to construct the continuum, and the way out by involving an "axiom of reducibility" demands an act of faith. In the formalist method everything hinges on whether it can be shown that the axioms used are free from contradiction, and recent work makes it unlikely that this can be proved within the system for the axioms of logic and arithmetic. The intuitionist method necessitates a rejection of part of received mathematics and a complication of the remainder; the existence of a mathematical entity may not be asserted until the entity has been constructed; a theorem like Fermat's last theorem might be neither true nor false, for it might, it is alleged, be impossible to construct its proof and also impossible to give an example of its falsity.

The main problems of the foundation of analysis centre round the properties of the natural numbers and of the continuum. The natural numbers, like the logical constants, are in a peculiar position for they are *used* in dealing with any set of axioms; even if the axioms of logic and arithmetic are formalised and emptied of meaning, the arguments employed in dealing with the symbols use an ordinary basic logic which is non-formalist and has a meaning, and a kind of finite arithmetic must also be used.

As soon as we are allowed to speak of *all* natural numbers (*e.g.* in the theorem, all natural numbers are sums of two, three, or four squares), the notion of an infinite set has been permitted in some sense, and if we are allowed to say that any set of natural numbers contains a least number, we have been permitted to speak of an arbitrary sub-class, finite or infinite, of an infinite class. But these apparently harmless liberties are circumscribed by the intuitionists.

More serious are the difficulties connected with the continuum, if a real number, given as the upper bound of a set of rationals, is to be regarded as the same sort of entity as one given as the upper bound of a set of previously given real numbers. On the status of the Aleph-numbers, there has never been agreement, some writers rejecting all after Aleph-null, while Hilbert defies anyone to drive him from the Cantorian paradise. But in spite of the controversies about the foundations of mathematics, there appears to be general agreement about much of the superstructure, and perhaps this shows that the proper policy would be to underpin rather than to rebuild; it is the difficulties

encountered by the logistician and the formalist in this attempt which have made some turn in despair to the intuitionist theory.

The first book under review gives a critical account of the *Principia*, of the formalist and intuitionist doctrines, and of the contributions of F. P. Ramsey, Weyl, Wittgenstein and Chwistik ; the discussion of the work of Brouwer and his school is particularly good and should remove much misunderstanding ; all readers who have seen Chwistik's papers will endorse the appeal our author makes to him for a connected account of his work and philosophical background.

No final account of any of the schools is yet possible, and this able interim report should be read by all interested in modern logic.

The second book, a competent introduction to the more elementary parts of logic, is much less original than the first. It is an interesting sign of the times that it gives an account not only of the traditional logic, but also of the simpler portions of the *Principia* treatment of propositions. There are chapters on induction which give an account of the theories of Mill and Keynes.

Now that logicians may be supposed familiar with the *Principia*, may we hope that they will soon take notice of the very interesting problems raised by modern mathematics. For example : given a set of axioms, when and how can their compatibility be proved ? If a theorem involves only the undefined entities used in the axioms, when can it be proved or disproved ? Why should some theorems be so much more difficult to prove than others not obviously deeper ? Is there any criterion of "depth" ? There is enough mathematics in existence now for it to be studied like the other works of man, and a discussion of the anatomy of demonstration would be infinitely more interesting than the puerilities which usually abound in books on logic.

H. G. F.

Principles of Geometry. V. Analytical principles of the theory of curves. Pp. x, 247. 15s. VI. Introduction to the theory of algebraic surfaces and higher loci. Pp. ix, 308. 17s. 6d. By H. F. Baker. 1933. (Cambridge)

Except, we believe, in Italy, the geometer is regarded as something of an oddity by his fellow mathematicians. It is not, we think, very difficult to give reasons for this. Modern geometry is a science which has not much in common with other branches of mathematics, and its technique is so specialised that it is difficult for anyone but a geometer to keep abreast with developments. And there are no possibilities of applications to induce the mathematical physicists, or the devotees of any other fashionable line of research, to take it up. A consequence of this is that geometers are to be found mostly congregated in little groups at a few centres. There the beginner learns the elements of his subject in lectures, and then becomes acquainted with the modern developments in advanced lectures, after which he begins reading original papers. The result is that there is little demand for treatises on modern geometrical theory, and hence the outside mathematical world is deprived of the means, even if it had the will, to know what the geometer is interested in. But sooner or later the demand for a carefully planned treatise cannot be ignored, and the time comes when it is impossible to rely entirely on original papers, which become more numerous every day and overlap and amend other papers until it becomes impossible to find one's way through the literature. Some attempt has indeed been made by the Italians to write up the subject, but their books show a certain reluctance to get beyond Volume I or II. The two present volumes, however, provide a successful answer to the first necessity of a treatise dealing with the basic notions of modern algebraic geometry.

Professor Baker and his school in Cambridge have devoted much study to the fundamental principles of algebraic geometry, and now he gives us a graphic account of his researches, and of the researches which he has inspired. When we have read these two volumes we feel that we have taken a great step

forward and the task of mastering the Italian researches on systems of curves on algebraic surfaces and on the classification of surfaces does not seem as overwhelming as it formerly appeared to the beginner. The ground is cleared, the notions are explained and a technique has been developed, and we feel more prepared for the task of arranging the theory in a logical order. The volumes in question are Volumes V and VI of the author's *Principles of Geometry*, which reached the fourth volume in 1925. In the preface to an earlier volume Professor Baker definitely stated that he did not intend to deal with the topics which form the subject of Volumes V and VI, but after eight years he has changed his mind, to the delight of his many followers. It might be argued that it would have been better to begin a new work, and, for instance, change to a type which is easier to the eyes, but there is a continuity in spirit with the earlier volumes of the *Principles*, and the new volumes have the same object as the earlier ones of describing and examining the basic principles of geometrical theory.

The special title of Volume V, "The Analytical Principles of the Theory of Curves", explains the general nature of the volume. The theory of curves has, of course, been the subject in many treatises. We feel, however, that Professor Baker has made a real contribution to geometrical literature in this volume. In it he has not confined himself to one point of view, and the geometrical, transcendental and arithmetic theories are all described. They are, indeed, more than described; the author has taken infinite pains to connect the various theories and to exhibit them as different view-points of the same question, and when we have read the volume we feel that we have coordinated the various theories as we have never done before. To those who are engaged in geometrical research this is invaluable, as they must be prepared at any time to use geometrical, algebraical, transcendental and even topological methods as the occasion arises, and to have the various methods examined and compared is a tremendous help. It leads, too, to greater precision, as, for instance, in the analysis of complicated singularities, which, though they may be properly understood in the case of curves, still lead sometimes to confusion in the case of higher loci.

The greater part of the volume is devoted to the examination of the principles of the various theories of the algebraic curve. These are first expounded and contrasted in detail, and then their importance is brought out by numerous and varied examples. These examples suggest to the reader all sorts of problems and one could spend endless time following up the ideas they suggest. There are, of course, certain applications of the general principles which for one reason or another have particular importance. Certain properties of postulation of twisted curves, for instance, are fundamental in the theory of surfaces. Such properties form the subject of the last chapter, whose object is to give the reader an extensive knowledge of the enumerative properties of curves which will stand him in good stead in the succeeding volume. There is little room for detailed criticism. The volume is too clearly the work of a master for that, and it is petty to complain of little idiosyncrasies such as writing g_r^n in place of the usual g_n^r to denote a linear series of sets of points on a curve of grade n and dimension r. Perfectly valid reasons are given for this, and opinions will always differ on the question of whether it is better to bow to convention or to make a change from the accepted notation for the sake of logic.

The first chapter of Volume VI—"Introduction to the Theory of Algebraic Surfaces and Higher Loci"—is a continuation of the theory of curves. The subject is the theory of correspondences of curves, and a detailed discussion is given of both the geometrical and transcendental aspects of the theory together with a description of related theories, such as that of defective integrals. Great pains, too, have been taken in the examples to show how the theory of correspondences can be applied with great effect to a large variety of problems

in the theory of curves, some of which the uninitiated might well think have no connection with correspondence theory. Consideration of these ideas and possible extensions of them leads one naturally to think of Schubert's correspondence principle, and this and related theories form the subject of Chapter II. The subjects of these two chapters are amongst the most fascinating in geometry, and we can never be indifferent to the power which the methods of correspondence theory give us in innumerable problems in algebraic geometry.

After these two chapters Professor Baker takes up the theory of the invariants of algebraic surfaces under birational transformation. A chapter on geometry in a plane, chiefly dealing with transformations and involutions, serves to introduce some useful notions and then a chapter dealing with some preliminary enumerative results for algebraic surfaces leads us to Chapter V, which deals with the basic notions in the theory of surfaces. The principal object of the chapter is to establish the existence of the fundamental invariants, p_g, p_n, I, ω, with, of course, the relation $I+\omega=12p_n+9$. These invariants are examined minutely, and the order of ideas to which each belongs is fully explained. Naturally it is impossible to deal with these ideas without coming up against the idea of exceptional curves, and the canonical system on the surface, and Professor Baker deals with these at length. The ideas involved here have played a fundamental part in the various researches which have been made in the theory of algebraic surfaces and the associated double integrals, and it is rather unfortunate that a certain amount of confusion has arisen between the various schools, chiefly owing to differences in definition. For instance, Nöther's definition of exceptional curves is somewhat different from that of the Italian geometers, and the fundamental theorem on the elimination of exceptional curves is not necessarily true for curves which Nöther calls exceptional. Professor Baker's critical examination does much to remove this confusion. Again there is room for more precision in the identification of the various definitions of the invariants when there are isolated singularities present on the surface. Professor Baker and some of his students have devoted much thought to this, and the results which they have obtained do a good deal towards removing some obscurities. The remaining chapters apply the basic notions to various problems, such as the intersection of loci, and certain surfaces which for various reasons have a special importance.

The object of this volume is therefore to put on a sound basis the theory of the fundamental invariants of an algebraic surface. With these dealt with adequately the subsequent researches on the geometry on a surface can be pursued more effectively, and much confusion can be avoided. But we must confess to a feeling of disappointment that Professor Baker has not seen fit to go more fully into the theory of birational transformations, which surely belongs to the same order of ideas. The problem of the transformation of a surface to one whose singularities are of the type called "ordinary" is discussed, but for the actual proof the reader is referred to original papers. Of the long list of papers referred to, very few are completely satisfactory, and we feel sure that Professor Baker could have made a valuable contribution to this fundamental problem. With this problem cleared up, and the discussion of the basic invariants which Professor Baker has given us, the difficulties of the researches which follow afterwards will be considerably lessened. Whether these are yet in a state to be finally written up is somewhat doubtful, and there is much truth in the suggestion made that topological considerations must play an important role in the final scheme. But it is by no means certain that we yet understand fully the bearing of topology on algebraic geometry. There are too many people working along different lines which have not yet converged, and until their various results are brought into line it will be difficult to assess exactly the extent to which topology will influence our subject.

W. V. D. H.

Algebraic Functions. By G. A. Bliss. Pp. ix, 218. $3.00. 1933. American Mathematical Society Colloquium Publications, 16. (American Mathematical Society)

English readers are apt to think of the Calculus of Variations in connection with Professor Bliss, and may be surprised to find him writing on Algebraic Functions. But those who have some acquaintance with American universities know that in recent years he has taken a keen interest in this subject, and has made some interesting contributions to it. With the cooperation of the American National Research Council he has now published as one of the Colloquium lectures the course which he has given several times in recent years in the University of Chicago. The course is of an introductory nature and deals with the arithmetical theory of algebraic functions, and the elementary theory of Riemann surfaces. No one could doubt that the book was written by an American, and, in our opinion, Americans are peculiarly successful in writing introductory text-books to branches of higher mathematics. The present book makes easy reading, and serves to give the beginner confidence in approaching the fascinating subject of Algebraic Functions. The principal questions of the function-theoretical approach to the subject are introduced in a simple straight-forward manner, and we feel that it would be difficult to improve on this as a first course on Algebraic Functions from the arithmetical point of view. Perhaps the most conspicuous feature of the book is the chapter on the reduction of singularities, but this is not surprising when we turn to the short bibliography at the end and notice that most of Professor Bliss's contributions have been on this topic. A novelty is the transformation of a curve to one with only double points in the "function-theoretical plane". This, in effect, amounts to a transformation into a curve in the projective plane (x, y, z) whose singularities are double points not on $z=0$, together with multiple points of orders r and s at $(1, 0, 0)$ and $(0, 1, 0)$ having distinct tangents which do not touch the curve elsewhere and are different from $z=0$, $(r+s)$ being the order of the curve. While this transformation of the curve may be convenient for the purposes of analysis we hardly feel that it is of great ultimate importance in the theory of curves. W. V. D. H.

Monografje Matematyczne. (Warsaw)

A series of monographs by members of the virile Polish school of mathematicians is very naturally devoted to the exposition of the most recent developments in the theory of functions of a real variable, since many of these developments are the fruits of the post-war revival in Polish mathematics, as everyone who has ever turned the pages of *Fundamenta Mathematicae* and *Studia Mathematica* knows.

The first of these volumes, by S. Banach, is reviewed below, and reviews of the two succeeding volumes, by S. Saks and by C. Kuratowski, will appear later; we are promised that monographs by W. Sierpinski and A. Zygmund are in preparation. The general editors have wisely decided that each volume must be written in either French, German, Italian or English.

Théorie des opérations linéaires. By S. Banach. Pp. viii, 254. $. 3. 1932. Monografje Matematyczne, 1. (Warsaw)

The many applications in real variable theory make this book of far greater interest to the modern analyst than the title might lead him to expect. The author gives us, not a "formal" generalisation, but a new method of attack for familiar problems.

These applications are mainly the work of the Polish school of mathematics. The ideas employed are extremely simple. We first choose a space in which the quantities we are concerned with are points and in which *distance* is suitably defined, as well as some form of the notion of *linear dependence*. To this space

we then apply familiar arguments, such as the famous *Stützgeradensatz* of Minkowski, which becomes in the hands of Banach a *continuation principle* similar to those of F. and M. Riesz. Again, a valuable *principle of condensation of singularities* is based on notions akin to "nearly nowhere" and "nearly everywhere" which Banach selects to be, not the straight generalisation of Lebesgue measure due to Daniell, Jessen and others, but the simpler notions due to Baire of sets of the first and of the second category. By this means we find, in special cases, that the "majority" of functions of a certain type exhibit some irregular behaviour and this often allows us to dispense with laborious constructions of special examples of such functions.

Applications are also made to the theory of measure, to convergence in mean for series and integrals, to orthogonal functions, to integral equations, to linear equations with an infinity of unknowns. In the course of the book very little knowledge is assumed, apart from the definition of the Lebesgue integral. L. C. Y.

Theorie der konvexen Körper. By T. Bonnesen and W. Fenchel. Pp. vii, 164. RM. 18.80. 1934. Ergebnisse der Mathematik, Band III, Heft 1. (Springer)

We have here an up-to-date and compact account of a theory which should form part of the equipment of every modern mathematician. It has applications to prime numbers, inequalities, differential geometry, hydrostatics, calculus of variations, linear functionals and even probability.

The proofs of the main theorems in the book are extremely elegant. In particular the famous *Stützgeradensatz* of Minkowski is proved in a few lines. On the other hand certain formal developments have been cut very short, doubtless for the sake of economy. The large list of references supplements this.

In the deduction of (4) and (5), § 37, from (3) and (8), § 36, the following remark will help the reader who is not expert at juggling with determinants: we first give to our integrand the form of that of (2), § 37. This is justified in the text if we are careful to observe that the case of linear dependence of certain tangential vectors is trivial. We then make use of the last equation of p. 59 and rewrite our integrand

$$\frac{\partial}{\partial\lambda}\left\{H\,.\,\|H_{\mu\nu}+\lambda\,\delta_{\mu\nu}\|\,.\,\|\xi,\frac{\partial\xi}{\partial\alpha_1},\ldots,\frac{\partial\xi}{\partial\alpha_{n-1}}\|\right\}_{\lambda=0}.$$

This leads at once to the formulae (4) and (5).

Among the results of special interest to the analyst we may mention: the minimum problems in the last part of the book and the inequalities of a geometrical character due to Brunn and Minkowski; the notion of *mixed volume*, a generalisation of both ordinary volume and area, about which the last word has not by any means been said, and which is connected with "relative" differential geometry and with probability; the notion of *sum* of convex figures which leads to that of mixed volume, and which, curiously enough, has been used by H. Bohr in connection with the Riemann ζ-function.

There are also interesting facts about centres of gravity related to Stieltjes integrals and hydrostatics, and there are some theorems of a purely real variable type asserting that angular points are enumerable and that bounded sequences of convex functions contain convergent subsequences. L. C. Y.

The Electromagnetic Field. By H. F. Biggs. Pp. viii, 158. 10s. 6d. 1934. (Clarendon Press)

This book is intended as a treatment of the classical theory of the electromagnetic field in a form suitable for students beginning an honours course in Physics. The equation giving the force acting on an electrically charged

particle moving relative to a magnetic field is taken as fundamental, and Maxwell's equations are deduced from it. The distribution of energy in the field is considered and the application of the theory to some particular problems is given. The last chapter is devoted to the Special Theory of Relativity and the electromagnetic field.

Vector methods are used throughout and the operations of vector algebra and analysis are introduced as required in the development of the physical theory, but with a complete lack of mathematical rigour. For example, in defining the *divergence* of a vector as the limit of the ratio of the flux out of a closed surface surrounding a point to the volume included, as the volume tends to zero, the proof that this limit is independent of the shape of the volume considered in proceeding to the limit is inadequate; the same applies to the definition of *curl*. Also, on p. 120, the reader is left with the impression that any set of nine scalar quantities forms a tensor.

The development of physical theory is in many places not clear, if not definitely misleading. This applies strongly to the treatment of the Larmor precession; no mention is made of the fact that the magnetic field is to be applied adiabatically, and it is therefore not clear what quantities are to be considered invariant as the field is applied. Another example is the unsatisfactory treatment of the theory of a dielectric. The existence of the potential and intensity at a point *in* a dielectric is assumed immediately without investigation, and the whole point of considering the field in a cavity of given shape appears to have been missed.

The book is not suitable for the class of students for which it is intended, and those at a more advanced stage will find much to criticise. B. S.

Wave Mechanics: Advanced General Theory. By J. Frenkel. Pp. viii, 525. 35s. 1934. (Oxford)

This book is the second of a series of three volumes on Quantum Mechanics; as its title implies, it contains an advanced mathematical discussion of the basis of the quantum theory; the applications of the theory are to be presented in a third volume.

It deals with matrix mechanics, transformation theory and the relation between wave mechanics and these more abstract aspects of atomic physics. However, the physicist, who may consult it, can rest assured that it is quite possible to understand and even to do research in the new mechanics without mastering its whole content; in fact, Professor Frenkel's first volume contains all the general theory that is necessary for the simpler applications. On the other hand, a knowledge of these advanced methods is essential for a proper understanding of the theory of the spinning electron and of relativistic quantum theory and also for an understanding of the very recent attempts of Heisenberg and Fermi to apply quantum theory to the nucleus.

This volume covers roughly the same ground as the famous treatise by Dirac, and indeed might profitably be read in conjunction with it, because, whereas Dirac's presentation of the theory is concise in the extreme, Professor Frenkel has illustrated the arguments by analogies drawn from other branches of physics, and, in this way, offers more assistance to the reader than does Dirac. At least half of the book is devoted to relativistic quantum mechanics, a subject in which considerable progress has been made during the last five years, and several of its latest developments are discussed at some length. It seems, however, surprising that there should be no mention of the very important work of Møller on the scattering of fast electrons.

One case of ambiguity in the book deserves comment, because it might lead to a false impression. The discussion of the neutron on page 348 makes it appear that the particle observed by Chadwick, Curie and Joliot, and christened by them the neutron, has some connection with the hypothetical "magnetic neutron", whose properties Dirac worked out, but which has never been

observed; or with the equally hypothetical "neutrino" (little neutron), whose existence Pauli has suggested to account for the disappearance of energy in β-decay. This, of course, is not the case, and theoretical physics, which has all but predicted the existence of the positive electron, cannot lay its claim to the neutron as well. N. F. M.

Tables for the Development of the Disturbing Function. By E. W. Brown and D. Brouwer. Pp. 69-157. 10s. 6d. 1933. (Cambridge)

A year ago when obituaries of Innes of Johannesburg were appearing on all sides, it was recalled that some forty or fifty years earlier he failed at the entrance examination for the Nautical Almanac Office—that initial slip was the prelude to a highly successful career in the service of Astronomy. Those who see in this a commentary upon our examinations system will be glad to note that the senior author of the book now under review sat for the examination (1891 April) for an Assistant at the Royal Observatory, Greenwich, but was not appointed.

Yet a little while and he became Professor of Applied Mathematics at Haverford College, Pennsylvania, and at once attacked the theory of the moon's motion, publishing his first results in 1897 and securing the Gold Medal of the Royal Astronomical Society in 1907. In the autumn of 1907 after spending sixteen years at Haverford College, E. W. Brown was appointed Professor of Mathematics at Yale University. From 1862 to 1922 our Nautical Almanac had derived its positions of the Moon from Hansen's *Tables de la Lune*, but in 1923 these were superseded by *Tables of the Motion of the Moon* by Ernest W. Brown, Yale University Press, 1919, 3 vols.

The volume of Tables under review is a separate edition, in book form, of Part V, Vol. 6, of the *Transactions of Yale University Observatory* and it supersedes Runkle, John D., *New Tables for determining the values of the coefficients in the perturbative function of planetary motion, which depend upon the ratio of the mean distances*, Washington, 1855. It will easily be seen that the volume under review embodies the experience of some forty years strenuous labour upon the moon, the motion of Jupiter's eighth satellite, the Trojan group of minor planets and other problems of interest in extremely difficult dynamical astronomy. Clearly this is a book for a very limited group of mathematical astronomers, so it may be sufficient to say that it is designed to economise labour in finding the coefficients of the various terms resulting from the harmonic analysis of the perturbing function. Frank Robbins.

Le Calcul Vectoriel : Cours d'Algèbre. By A. Véronnet. With a preface by H. Villat. Pp. xviii, 252. 50 fr. 1933. (Gauthier-Villars)

In this book M. Véronnet, who performed a useful task some years ago in translating Coffin's *Vector Analysis* into French, has a double aim. First, he shows how by sheer interpretation of symbols in different ways simple algebraical identities may be made to carry the weight of immensely complicated relations between polynomials; he emphasizes both the plurality of algebras and the variety of applications of an individual algebra. Secondly, he develops the vectorial algebra and analysis of Euclidean three-dimensional space, as an illustration of general methods, in a manner to form a direct introduction to the calculus of tensors, so that he is able to come naturally to the spaces of Riemann and Cartan.

With such a programme, it follows that there is a great deal of excellent matter in the book. There is perhaps less novelty than M. Villat's preface might lead the reader to suppose. The use of Grassmann's alternate units for the development of the theory of determinants has been familiar in this country since the publication of Scott's treatise in 1880. That the key to the manipulation of oblique axes is in a free use of the reciprocal frame, which

restores to the formulae the simplicity which they have when the axes are rectangular, is also a commonplace here.

But that M. Véronnet has been discovering for himself what he might have been learning from others is not very important, and does not diminish the infection of his enthusiasm or the merits of his exposition. The defect of the book is that he is one of those mathematicians, now very rare, who are alive to the beauties of algebra but insensitive to the subtleties of analysis. How can we take seriously a writer who tells us that whereas the rational numbers form a linear continuum which is not uniformly dense, the representation of p/q by the point (p, q) distributes the rational numbers uniformly over the plane, and the further passage to the point where the radius to (p, q) cuts the unit circle gives a representation on the circumference, which is continuous, uniform, and linear? How can we risk putting into confiding hands a course in which a result written in the form

$$\left(1+\frac{1}{m}\right)^m=1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{3!}+\ldots+\frac{1}{p!}+\ldots=e \quad (m=\infty)$$

is proved in three lines by the simple consideration that as m tends to infinity, the ratios m/m, $m(m-1)/m^2$, etc. tend to unity?

For M. Véronnet's mastery of pure algebra and for his comprehensive outlook there can be nothing but admiration. It is a matter for sincere regret that this book is adulterated with such questionable analysis that it cannot be recommended except to mathematicians whose grasp of first principles is absolutely secure.

E. H. N.

Caliban's Problem Book. By H. Phillips, S. T. Shovelton, and G. S. Marshall. Pp. x, 330. 6s. 1933. (De La Rue)

If a review is belated, the reason may be adequate, but it is seldom creditable to everybody concerned. This notice is the welcome exception; it has been held up deliberately, for what is the use of recommending a book of problems to a schoolmaster in February or May? But in July, yes! Whether it is the family trunk or the independent haversack that is to be packed, to find room for Caliban is to insure against tedium if bad weather keeps you indoors, and to provide a better occupation out of doors than the invention of targets for material pebbles.

To attempt to classify these problems and enumerate the types would be absurd. There are cryptograms, problems in arrangement and elimination, problems leading to indeterminate equations, problems in probability and detection. The gem of the collection is certainly Mr. Newman's problem in pure logic, which must be solved to be believed:

> When Caliban's will was opened it was found to contain the following clause:
>
> "I leave ten of my books to each of Low, Y. Y., and 'Critic', who are to choose in a certain order.
>
> No person who has seen me in a green tie is to choose before Low.
>
> If Y. Y. was not in Oxford in 1920 the first chooser never lent me an umbrella.
>
> If Y. Y. or 'Critic' has second choice, 'Critic' comes before the one who first fell in love."
>
> Assuming the problem to contain no statement superfluous to its solution, *What was the prescribed order of choosing; and who lent Caliban an umbrella?*

Detailed solutions of the 105 problems occupy 172 pages, and a short appendix includes tables of squares and prime numbers, and some notes on properties of numbers and on cyclic elimination. A numbered alphabet would be an equally useful tool to add in a future edition.

It must not be thought that the kind of numerical questions on which solutions of many of these problems depend is never encountered in real mathematical life. On the contrary, the construction of exercises in algebraical geometry and in mechanics with rational solutions often depends on questions of this kind. For example, in looking for a family of cubic curves with three rational singularities, I found that I needed rational numbers k, q, such that

$$1-4(k+1)(k-1)^3/k(2k-1)^3(1-q^2)$$

should be a non-trivial perfect square, a requirement as artificial in appearance as any to which Caliban problems lead, and a strong inducement, therefore, to leave my work and sharpen my wits at play.

There is a superfluous datum in problem 81, and a misprint in problem 75. The aesthetic fault, fortunately, has only the most negligible bearing on the reconstruction of the score-sheet of the Insect League, but a reader who noticed the letters *EAD* in a cell on p. 94 before he had gone so far in reading the diagram as to be confident that he was using the right method would infer at once that the right method must be wrong; the letters should be *END*. An appropriate conclusion for me, but the lively enunciations raise another problem. Is the writer who puts Moth Eaton on the map and Graeme Atta into a professorial chair the man who puts Sheila Daw on the hoardings? It is an awful responsibility.

Don't forget to pack scribbling paper. E. H. N.

Numerical Trigonometry and Mensuration. By W. J. Walker. With Four-figure tables. Pp. 108, 14, xii. 3s. Tables alone, 6d.; paper cover, 4d. 1932. (T. Victor White)

Trigonometry for Schools. By F. J. Hemmings and J. F. Chalk. Pp. viii, 266. 4s. 6d. 1933. (Blackie and Son)

A good many text-books on elementary mathematics seem nowadays to owe their existence not to any burning conviction on the part of the authors that their experience and mental equipment peculiarly fit them to represent these elements from a new angle or with especial clarity, as to the desire to reproduce familiar material in a form specially adapted for absorption by students of a particular age or preparing for a particular class of examination. Thus the first of the above books is expressly designed for pupils who are beginning Trigonometry at the age of 14 or 15 and are preparing for examinations of the standard of the Oxford School Certificate (for which, as we are informed on the title-page, the author is an examiner). As he explains in the preface, Mr. Walker considers that for beginners of this age, the usual modern practice of presenting the ratios one by one is unnecessary. So we have the sine, cosine, tangent, introduced on page 1 and the other three on page 5. Some teachers who favourably incline to this view may nevertheless feel that a more comprehensive set of examples on the first three ratios should have preceded the second group. The text follows the usual lines of "numerical Trigonometry". There are chapters on mensuration and on surveying. The algebraic manipulation of the ratios is scarcely touched upon except in the short appendix. The best parts of the book are the examples; these are numerous and varied. The text is over-compressed and rather gives the impression of a longer book cut down to fit into 100 pages. It is curious that the author does not mention the term coordinates in the chapter on the "general angle". Although the text of this chapter might not cause searchings of heart to the average pupil, an intelligent one with a logical mind might ask some awkward questions, as, for instance, when an angle greater than 90° can only be considered as obtained by counter-clockwise rotation from the line OX, why an acute angle in the same figure can apparently be obtained by rotation in either sense from the line OX', and how a line ON can be positive and negative at the same time.

The chapters on mensuration are included, according to the preface, "not only for the sake of candidates who are taking the joint subject in such examinations as the Oxford School Certificate, but also because of the many applications of Trigonometry which it affords". One of the difficulties in teaching mensuration is that some of the formulae (*e.g.* for the volume of a cone and a sphere) cannot easily be demonstrated without the use of Calculus methods. But many others can be so proved. In this book we find a mere collection of formulae "for convenience of reference" without any attempt at demonstration. It surely cannot be the author's intention that such a formula as $\frac{1}{3}(A+\sqrt{AA'}+A')H$ for the volume of the frustum of a cone should be memorised, quoted, and used, without the pupil being taken through a proof of it. For that would be not education but cram.

There are errors in one or two unessential points. A nautical mile is given as 6083 feet instead of 6080. There is an oddly worded note on Points of the Compass. "... directions such as NE, NW, SE, SW. The sailor further bisects these angles of 45° and uses such directions as NNE—halfway between N and NE, and so on; he even bisects them again, but in Trigonometry it is more convenient to name directions in another way. Thus in Fig. 40 the line OA is said to point in the direction 55° N of E (or E 55° N) ... Of course OA could have been described as N 35° E." In point of fact, the description of courses and bearings by "points" of the Mariner's Compass is, to all intents and purposes, obsolete, though NE, NW ... are occasionally used. The sailor and airman measure courses and bearings in degrees from the North and South line, or, by the gyro-compass method, from 0 to 360° clockwise from North. Of course, "in Trigonometry" it is open to a composer of examples to name bearings in any unambiguous way he chooses, but it seems perverse to use E 55° N when every practical man uses N 35° E, and the former is no easier to understand than the latter.

Some of the figures in the book are carelessly drawn and have not printed well. The sine curve on p. 95 shows sin 0 as 0·05, and the portion of the curve from 0 to 50° looks as though it had been ruled with a straight-edge. As a compensation, the sine curve on p. 18 cuts the angle-axis at 2°. The labelling of figures such as 50, 85 and 119 is confusing. Bound up with the book are four-figure tables of logarithms, antilogarithms, natural ratios, and logarithmic sines, cosines, tangents and cotangents. The usual practice of five and six-figure table books is adopted of using one table for sines and cosines, etc., the degrees and minutes for the co-functions being shown along the bottom and up the right-hand side of the page. This is a sensible practice and saves turning over pages; one wonders why it is not generally adopted. Of course, a table of antilogarithms is really a luxury, and it is doubtful whether the use of such a table does not cause more mistakes than it saves.

Trigonometry for Schools is the antithesis of the last book. The text is attractively and carefully written but is unnecessarily spun out. It could have been compressed without loss and so room could have been found for more examples. The ratios are introduced singly. Then follow chapters on algebraical development. It is an unusual procedure to obtain formulae for $3A$ (using a projection method—in fact that for $\sin(A+B)$ in a special case) before $\sin(A+B)$ has been proved. The more general cases of the compound angle formulae follow. So far, only acute angles have been dealt with. Then follows a longish chapter on the complete graphs of the ratios in which the sines, etc., of angles greater than 90° are obtained by use of the addition formulae. Logically this does not get anywhere, because it has to be assumed that the $\sin(A+B)$ formula holds for angles greater than 90°, whereas the sine and cosine for such angles have not yet been defined. The justification of this lies in the continuity of the graphs, but it does not seem that much has been gained. The complete definition of the ratios by coordinates is carefully done in the next chapter. There is a chapter on logarithms—surely unnecessary in

these days when they are normally introduced in connection with arithmetic or algebra. There follow chapters on solution of triangles, area of triangle, circles connected with the triangle. As remarked above, the book is rather deficient in examples. In particular, the set on "heights and distances" is meagre. H. E. P.

Recent Developments in the Teaching of Geometry. By J. Shibli. Pp. viii, 252. $2.25. 1932. (State College, Pennsylvania)

This book attempts to answer the question whether geometry as an educational subject has been progressive and to investigate the important developments in the teaching of elementary geometry especially since the beginning of the present century. The author has brought together and arranged in a most interesting manner a very large amount of information as to methods and syllabus of teaching, mainly in the United States, but also with constant reference to what has been and is being done in other countries. It is instructive to note that he regards the 1923 Report of the Mathematical Association on *The Teaching of Geometry in Schools* as representing the best British practice, and quotes with approval from this document in chapter after chapter.

To quote from the preface: "Ch. I gives a general review of the historical development of the teaching of geometry. Ch. II considers the factors that have exerted a marked influence on the teaching of geometry. Ch. III discusses the development of intuitive geometry (the Stage A geometry of the 1923 Report) and its teaching in the junior high school. The remaining chapters trace the recent developments in the purpose of demonstrative geometry, in its content, and in the method of teaching it. The book discusses the problems that teachers of geometry are facing at present and the various attempts that are being made to solve them".

The chapter on the foundations of geometry, Ch. V, is especially well arranged and contains in an attractive form about as much information as to postulates, axioms, etc., as is likely to be of interest or use to any teacher who does not specialize in this section.

The chapter on "The propositions of plane geometry", besides giving an elaborate analysis of the number of propositions included in text-books of various dates, and the complete list of the propositions recommended by the National Committee and of those which its list discards from the most favoured list of the beginning of the century, contains a most interesting feature in a collection of proofs of the angle-sum-of-triangle theorem as printed in popular text-books of various dates.

Ch. VIII, on "The relation of plane geometry to other subjects", discusses *inter alia* "the combination of plane and solid geometry" at considerable length—a subject to the forefront at the present time in discussions between teachers of geometry in this country. Perhaps the most interesting part of Ch. IX, on the "Aims of demonstrative geometry", is the note as to the latest views of psychologists on the "transfer of training" with the conclusion that "a majority of the psychologists seem to believe that, with certain restrictions, transfer of training is a valid aim in teaching".

Two short chapters on "The new emphasis in geometry" and a summary conclude a most interesting book, well worth the careful attention of thoughtful teachers. C. O. Tuckey.

School Certificate Algebra. By D. Larrett. Pp. 258 + 135. With answers, 5s. Without answers, 4s. 6d. 1934. (Harrap)

This volume consists of the author's *Junior Algebra*, reviewed p. 146, vol. xvi (May, 1932), and parts of his *Senior Algebra*, reviewed p. 350, vol. xvii (December, 1933). In these reviews the get-up, the painstaking exposition and other good qualities were commended; while quotations illustrating the

author's attitude to certain parts of the subject were made with little comment. Members of the Association would recognize from the quotations that the author's methods were not always those which the Reports of the Association would approve. In the volume under review these methods are retained, as are certain pieces of English unintelligible in themselves and insulting to the intelligence of the pupil who is in a position to dispense with them. Thus on p. 27, "We can multiply a number by itself as many times as we please, the index showing the number of terms in the product", and p. 92, "When a statement is written in a short form and consists of two equal parts it is called an equation".

In the chapter on Directed Number he writes (p. 133)

$$(+4)\times(+3)=(+4)+(+4)+(+4),$$

making it clear that he is defining $(+4)\times(+3)$ as repeated addition of $(+4)$. On the next pages he defines $(-4)\times(-3)$ in set terms as a repeated subtraction and proceeds

$$\begin{aligned}(-4)\times(-3)&=-(-4)-(-4)-(-4)\\&=-(-12)\\&=+12.\end{aligned}$$

"We thus obtain the following rule for the product of two directed numbers . . .".

Mr. Larrett may maintain his right to deal with the solution of equations in his own ingenious way, in the face of the received ideas of to-day and the recommendations of the Association's Reports, but his ingenuity in eluding the difficulty of multiplying directed numbers is fraudulent. No doubt the pupil will be satisfactorily deceived, but, if Mr. Larrett shares the deception, we recommend him to study carefully the sections on Directed Number in the forthcoming *Report on the Teaching of Algebra*. F. C. B.

Some great mathematicians of the nineteenth century. II. By G. Prasad. Pp. xviii, 324. RM. 6. 1934. (Benares Mathematical Society; Koehlers Antiquarium, Leipzig)

The second volume of Professor Prasad's work deals with Cayley, Hermite, Kronecker, Brioschi, Cremona, Darboux, Cantor, Mittag-Leffler, Klein and Poincaré. The method is similar to that of the first volume, a succinct biography mingled with an account of the most important publications of the mathematician in question. The design presents even more difficulties to an author than it did in the first volume, for to deal with ten such men in 320 not very large pages demands conciseness of exposition and clarity of style combined with unerring discrimination of those of a mathematician's writings which are really vital to a right understanding of his position and influence. Professor Prasad has made a most praiseworthy attempt to cope with these difficulties; it is not the least of his virtues that he leaves us anxious to renew and extend our acquaintance with the men he mentions.

Professor Prasad in picking his team must have felt like a Test Match selector, whose problem is doubtless frequently "Can I possibly leave X out?" It is clear that "Mathematicians" in the title means "pure mathematicians", which accounts for the neglect of the great galaxy of mathematical physicists of the nineteenth century, and in particular of Maxwell. But after that it is not completely obvious how the canon has been framed. Not only a mathematician's publications but also his influence on younger men must have been taken into account, since, for example, if we feel that Hamilton, say, was a more brilliant figure than Brioschi, we must remember that it is to Brioschi and to his pupil, Cremona, that we mainly owe the brilliant Italian school of recent times.

It would not be difficult to swell this review into a substantial article by commenting on the men Professor Prasad has chosen, or by yielding to the

temptation to stress the importance of some of the mathematicians whose claims have had to be ignored. But such an expansion would not serve our purpose, which is to recommend this book very warmly to all those who wish to get clear and accurate ideas of the main lines of mathematical progress during the nineteenth century, as well as some concrete knowledge of the leading figures; perhaps even this recommendation is superfluous in view of Professor Watson's remarks on Volume I of this treatise in his Presidential Address (see p. 6 of this volume of the *Gazette*).

Any school library which does not procure Professor Prasad's book is being mismanaged.

T. A. A. B.

Grundlagen der Analysis. By E. Landau. Pp. xiv, 134. Geh. RM. 7.70. Geb. RM. 8.80. 1930. (Akademische Verlagsgesellschaft, Leipzig)

Einführung in die Differentialrechnung und Integralrechnung. By E. Landau. Pp. 368. Geh. RM. 20. Geb. RM. 22.50. 1934. (Noordhoff, Groningen)

The review of the first of these books is not so belated as it may seem, for on reading the second it became obvious that the two ought to be considered together and as the *Grundlagen* had not been received when published, a request to the publishers met with a courteous reply in the shape of a review copy.

We have then, in these two rather expensive but superbly produced volumes, a formulation of the groundwork of mathematical analysis, written in that incisive and pregnant style of which Professor Landau is a master. He is no expositor of "Mathematics made easy" or "Mathematics without tears"; he is not writing for the dull or mediocre student, but for the genuine mathematician in the making, who will eventually recognise that Professor Landau's style, whether he likes it or not, is one that he must appreciate in order to read the present-day periodical literature of analysis. It is, of course, not difficult to write on analysis with an economy of words like that of Professor Landau; the difficulty is to avoid the genesis of jerks and discontinuities which break the general thread of the exposition; it need hardly be said that with Professor Landau we rarely feel that continuity has been sacrificed to brevity.

The first book deals with integers, fractions, real numbers and complex numbers. We start with an account of the integers, based on a set of axioms which are essentially those given by Peano; then the fractions follow, treated of course in a logical and abstract fashion. The next two chapters discuss the Dedekind section and the real number; while it is enough that a student should have a general idea of the work on integers and fractions and that he should know where to lay his hands on an account of it, the theory of the real number is different in character and in importance, and Professor Landau does not attempt to cover the whole ground in the fifty pages occupied by these two chapters, but rather seeks to deal exhaustively with the definition, addition and multiplication of real numbers, leaving questions of powers and roots (apart from the square root) for further consideration in the second book. The final chapter expounds the arithmetic theory of the complex number.

In the second book, starting from the foundations laid in his *Grundlagen*, Professor Landau erects thereon the structure of the calculus, as a pure mathematician sees it, as far as the elements of Fourier series and the Gamma function. There are no examples, and no applications to other branches of mathematics. The volume is definitely meant for the fairly sophisticated Honours student with an enthusiasm for mathematics as an end in itself, and is not suitable for any other type of student; for instance, the definite integral is a number which, under certain conditions, can be associated with a function defined in some interval, and there is no mention of its geometrical significance, not because the author doubts its importance, but because he is writing a rigorously logical treatise and to treat the geometrical applications of the

calculus on an equal footing would involve writing another book as large as the present one.

Space thus saved is used, for example, to provide a climax to the chapter on general properties of continuous functions by a proof of Weierstrass' theorem on polynomial approximations to a continuous function, to include immediately after the definition of differentiability an ingenious construction due to van der Waerden of an everywhere continuous, nowhere differentiable function.

It is surely to be regretted that Professor Landau has not been able to adopt Professor Hardy's method of introducing the logarithm; presumably he thinks it desirable to have defined the logarithmic and exponential functions before setting up the machinery of the calculus, and to that end he deals in his first chapter with sequences and their limits and then in the second defines the logarithm of a positive x as

$$\lim_{n \to \infty} k(\sqrt[k]{x}-1), \text{ where } k=2^n ;$$

the exposition is ingenious and, of course, logically perfect, but it is somewhat artificial and for that reason not very convincing. And probably the exposition of the theory of uniform convergence would be more easily grasped by the reader if the concept were exhibited as a property relating to the structure of sequences rather than as an *ad hoc* device for ensuring the continuity of the sum-function of an infinite series.

But after all it is presumptuous to differ from Professor Landau, while it is almost certainly superfluous to recommend these two books to teachers of analysis, for they will not need to be assured that the account which this great master has given us of the elements of his subject is, like his earlier classical treatises, a happy compound of clarity, force and logic. T. A. A. B.

A Bibliography of George Berkeley. By T. E. Jessop. With an inventory of Berkeley's manuscript remains by A. A. Luce. Pp. xvi, 99. 7s. 6d. 1934. (Oxford)

Most of us, no doubt, know Berkeley only by some remembrance of quips about "ghosts of departed quantities" and of a philosophic theory which Samuel Johnson thought to have demolished by kicking a stone, an operation which, we feel pleasantly certain, hurt Johnson's foot more than it did Berkeley's theory. But Cajori will tell us that the publication—just 200 years ago—of *The analyst; or, a discourse addressed to an infidel mathematician. Wherein it is examined whether the object, principles, and inferences of the modern analysis are more distinctly conceived, or more evidently deduced, than religious mysteries and points of faith*, was "the most spectacular mathematical event of the eighteenth century in England": and the words of A. J. Balfour provide a measure of Berkeley's intellectual stature: "scarce any man of his generation touched contemporary life at so many points. In reading his not very voluminous works we find ourselves ... in the thick of every great controversy—theological, mathematical, and philosophical—which raged in England during the first half of the eighteenth century".

Professor Jessop has compiled a most scholarly bibliography of Berkeley, listing collected works, works published by Berkeley, posthumously published remains, translations and some 400 references to works and articles on Berkeley. Excellently produced by the Oxford press, it is obviously an admirable piece of research apparatus.

It is interesting to observe that Berkeley's first publication bears the title *Arithmetica absque algebra aut Euclide demonstrata. Cui accesserunt, cogitata nonnulla de radicibus surdis, de aestu aeris, de ludo algebraico, &c.* In view of Berkeley's later performances, it is surprising to find, on referring to the essays themselves, that they are overwhelmingly dull. T. A. A. B.

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